

DYNAMIC OF THRESHOLD SOLUTIONS FOR ENERGY-CRITICAL WAVE EQUATION

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ABSTRACT. We consider the energy-critical non-linear focusing wave equation in dimension $N = 3, 4, 5$. An explicit stationary solution, W , of this equation is known. In [KM06b], the energy $E(W, 0)$ has been shown to be a threshold for the dynamical behavior of solutions of the equation. In the present article we study the dynamics at the critical level $E(u_0, u_1) = E(W, 0)$ and classify the corresponding solutions. We show in particular the existence of two special solutions, connecting different behaviors for negative and positive times. Our results are analogous to [DM07], which treats the energy-critical non-linear focusing radial Schrödinger equation, but without any radial assumption on the data. We also refine the understanding of the dynamical behavior of the special solutions.

1. INTRODUCTION AND MAIN RESULTS

We consider the focusing energy-critical wave equation on an interval I ($0 \in I$)

$$(1.1) \quad \begin{cases} \partial_t^2 u - \Delta u - |u|^{\frac{4}{N-2}} u = 0, & (t, x) \in I \times \mathbb{R}^N \\ u|_{t=0} = u_0 \in \dot{H}^1, \quad \partial_t u|_{t=0} = u_1 \in L^2. \end{cases}$$

where u is real-valued, $N \in \{3, 4, 5\}$, and $\dot{H}^1 := \dot{H}^1(\mathbb{R}^N)$. The theory of the Cauchy problem for (1.1) was developped in many papers (see [Pec84, GSV92, LS95, SS94, SS98, Sog95, Kap94]). Namely, if $(u_0, u_1) \in \dot{H}^1 \times L^2$, there exists an unique solution defined on a maximal interval $I = (-T_-(u), T_+(u))$ and the energy

$$E(u(t), \partial_t u(t)) = \frac{1}{2} \int |\partial_t u(t, x)|^2 dx + \frac{1}{2} \int |\nabla u(t, x)|^2 dx - \frac{1}{2^*} |u(t, x)|^{2^*} dx$$

is constant ($2^* := \frac{2N}{N-2}$ is the critical exponent for the H^1 -Sobolev embedding in \mathbb{R}^N).

An explicit solution of (1.1) is the stationary solution in \dot{H}^1 (but in L^2 only if $N \geq 5$)

$$(1.2) \quad W := \frac{1}{\left(1 + \frac{|x|^2}{N(N-2)}\right)^{\frac{N-2}{2}}}.$$

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The works of Aubin and Talenti [Aub76, Tal76], give the following elliptic characterization of W (throughout the paper we denote by $\|\cdot\|_p$ the L^p norm on \mathbb{R}^N)

$$(1.3) \quad \forall u \in \dot{H}^1, \quad \|u\|_{2^*} \leq C_N \|\nabla u\|_2$$

$$(1.4) \quad \|u\|_{2^*} = C_N \|\nabla u\|_2 \implies \exists \lambda_0 > 0, x_0 \in \mathbb{R}^N, \delta_0 \in \{-1, +1\} \quad u(x) = \frac{\delta_0}{\lambda_0^{(N-2)/2}} W\left(\frac{x+x_0}{\lambda_0}\right),$$

where C_N is the best Sobolev constant in dimension N .

The dynamical behavior of some solutions of (1.1) was recently described in [SK05], [SKT07] (in the radial three-dimensional case) and [KM06b]. In [KM06b], Kenig and Merle has shown the important role of W , whose energy $E(W, 0) = \frac{1}{NC_N}$ is an energy threshold for the dynamics in the following sense. Let u be a solution of (1.1), not necessarily radial, such that

$$(1.5) \quad E(u_0, u_1) < E(W, 0).$$

Then

- if $\|\nabla u_0\|_2 < \|\nabla W\|_2$, we have $I = \mathbb{R}$ and $\|u\|_{L_{t,x}^{\frac{2(N+1)}{N-2}}} < \infty$, which implies from the linear theory of (1.1) that the solution scatters;
- if $\|\nabla u_0\|_2^2 > \|\nabla W\|_2^2$ then $T_+ < \infty$ and $T_- < \infty$.

Our goal (as is [DM07] for the nonlinear Schrödinger equation in the radial case) is to give a classification of solutions of (1.1), not necessarily radial, with *critical* energy, that is with initial condition $(u_0, u_1) \in \dot{H}^1 \times L^2$ such that

$$E(u_0, u_1) = E(W, 0).$$

The stationary solution W belongs to this energy level, is globally defined and does not scatter. Another example of special solutions is given by the following.

Theorem 1 (Connecting orbits). *Let $N \in \{3, 4, 5\}$. There exist radial solutions W^- and W^+ of (1.1), with initial conditions $(W_0^\pm, W_1^\pm) \in \dot{H}^1 \times L^2$ such that*

$$(1.6) \quad E(W, 0) = E(W_0^+, W_1^+) = E(W_0^-, W_1^-),$$

$$(1.7) \quad T_+(W^-) = T_+(W^+) = +\infty \text{ and } \lim_{t \rightarrow +\infty} W^\pm(t) = W \text{ in } \dot{H}^1,$$

$$(1.8) \quad \|\nabla W^-\|_2 < \|\nabla W\|_2, \quad T_-(W^-) = +\infty, \quad \|W^-\|_{L^{\frac{2(N+1)}{N-2}}((-\infty, 0) \times \mathbb{R}^N)} < \infty,$$

$$(1.9) \quad \|\nabla W^+\|_2 > \|\nabla W\|_2, \quad T_-(W^+) < +\infty.$$

Remark 1.1. Our construction gives a precise asymptotic development of W^\pm near $t = +\infty$. Indeed there exists an eigenvalue $e_0 > 0$ of the linearized operator near W , such that, if $\mathcal{Y} \in \mathcal{S}(\mathbb{R}^N)$ is the corresponding eigenfunction with the appropriate normalization,

$$(1.10) \quad \|\nabla (W^\pm(t) - W \pm e^{-e_0 t} \mathcal{Y})\|_{L^2} + \|\partial_t (W^\pm(t) - W \pm e^{-e_0 t} \mathcal{Y})\|_{L^2} \leq C e^{-2e_0 t}.$$

We refer to (6.1) and (6.9) for the development at all orders in $e^{-e_0 t}$.

Remark 1.2. Similar solutions were constructed for NLS in [DM07]. However, in the NLS case, we were not able to prove that $T_-(W^+) < \infty$ except in the case $N = 5$. We see this fact, in particular in the case $N = 3$, as a nontrivial result. Note that W^+ is not in L^2 except for $N = 5$, so that case (c) of Theorem 2 below does not apply.

Our next result is that W , W^- and W^+ are, up to the symmetry of the equation, the only examples of new behavior at the critical level.

Theorem 2 (Dynamical classification at the critical level). *Let $N \in \{3, 4, 5\}$. Let $(u_0, u_1) \in \dot{H}^1 \times L^2$ such that*

$$(1.11) \quad E(u_0, u_1) = E(W, 0) = \frac{1}{NC_N^N}.$$

Let u be the solution of (1.1) with initial conditions (u_0, u_1) and I its maximal interval of definition. Then the following holds:

- (a) *If $\int |\nabla u_0|^2 < \int |\nabla W|^2 = \frac{1}{C_N^N}$ then $I = \mathbb{R}$. Furthermore, either $u = W^-$ up to the symmetry of the equation, or $\|u\|_{L_{t,x}^{\frac{2(N+1)}{N-2}}} < \infty$.*
- (b) *If $\int |\nabla u_0|^2 = \int |\nabla W|^2$ then $u = W$ up to the symmetry of the equation.*
- (c) *If $\int |\nabla u_0|^2 > \int |\nabla W|^2$, and $u_0 \in L^2$ then either $u = W^+$ up to the symmetry of the equation, or I is finite.*

The constant C_N is defined in (1.3). In the theorem, by u equals v up to the symmetry of the equation, we mean that there exists $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^N$, $\lambda_0 > 0$, $\delta_0, \delta_1 \in \{-1, +1\}$ such that

$$u(t, x) = \frac{\delta_0}{\lambda_0^{(N-2)/2}} v\left(\frac{t_0 + \delta_1 t}{\lambda_0}, \frac{x + x_0}{\lambda_0}\right).$$

Remark 1.3. Case (b) is a direct consequence of the variational characterization of W given by (1.4). Furthermore, using assumption (1.11), it shows (by continuity of u in \dot{H}^1 and the conservation of energy) that the assumptions $\int |\nabla u(t_0)|^2 < \int |\nabla W|^2$, $\int |\nabla u(t_0)|^2 > \int |\nabla W|^2$ do not depend on the choice of the initial time t_0 . Of course, this dichotomy does not persist when $E(u_0, u_1) > E(W, 0)$.

Remark 1.4. Theorem 2 is also the analogous, for the wave equation, of Theorem 2 of [DM07] for NLS, but without any radial assumption. The nonradial situation carries various problems partially solved in [KM06b], the major difficulty being a sharp control in time of the space localization of the energy. We conjecture that the NLS result also holds in the nonradial situation. Note that case (a) implies (W being radial) that any solution of (1.1) satisfying (1.11) and whose initial condition is not radial up to a space-translation must scatter if $\int |\nabla u_0|^2 < \int |\nabla W|^2$.

Remark 1.5. In dimension $N = 3$ or $N = 4$, W^+ is not in L^2 , and case (c) means that any critical-energy solution such that $\int |\nabla u_0|^2 > \int |\nabla W|^2$ and $u_0 \in L^2$ blows-up for $t < 0$ and $t > 0$. It seems a delicate problem to get rid of the assumption $u_0 \in L^2$.

Remark 1.6. As a corollary, in dimension $N = 5$, a dynamical characterization of W is obtained. It is, up to the symmetry of the equation, the only L^2 -solution such that $E(u_0, u_1) \leq E(W, 0)$ that does not explode and does not scatter neither for positive nor negative time.

The paper is organized as follows. In Section 2 we recall previous results on the Cauchy Problem for (1.1) and give preliminary properties of solutions of (1.1) at the energy threshold such that $\int |\nabla u_0|^2 < \int |\nabla W|^2$ and which do not scatter for positive times. These properties

mainly follow from [KM06b]. In Section 3, we show that these solutions converge exponentially to W as $t \rightarrow +\infty$, which is the first step of the proof of Theorem 2 in case (a). In Section 4, we show the same result for energy-threshold solutions such that $\int |\nabla u_0|^2 > \int |\nabla W|^2$, $u_0 \in L^2$ and that are globally defined for positive time. In Section 5, we study the linearized equation around the solution W . Both theorems are proven in Section 6. The main tool of the proofs is a fixed point giving the existence of the special solutions and, by the uniqueness property, the rigidity result in Theorem 2.

2. PRELIMINARIES OF SUBCRITICAL THRESHOLD SOLUTIONS

2.1. Quick review on the Cauchy problem. We recall some results on the Cauchy Problem for (1.1). We refer to [KM06b, Section 2], for a complete overview. If I is an interval, write

$$(2.1) \quad S(I) := L^{\frac{2(N+1)}{N-2}}(I \times \mathbb{R}^N), \quad N(I) := L^{\frac{2(N+1)}{N+3}}(I \times \mathbb{R}^N)$$

$$(2.2) \quad \|u\|_{\ell(I)} := \|u\|_{S(I)} + \|D_x^{1/2}u\|_{L^{\frac{2(N+1)}{N-1}}(I \times \mathbb{R}^N)} + \|\partial_t D_x^{-1/2}u\|_{L^{\frac{2(N+1)}{N-1}}(I \times \mathbb{R}^N)}.$$

We first consider the free wave equation:

$$(2.3) \quad \partial_t^2 u - \Delta u = f, \quad t \in (0, T)$$

$$(2.4) \quad u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1,$$

where $D_x^{1/2}f \in N(0, T)$, $u_0 \in \dot{H}^1$, $u_1 \in L^2$. The solution of (2.3), (2.4) is given by

$$u(t, x) = \cos(t\sqrt{-\Delta})u_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}u_1 + \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}f(s)ds.$$

Then we have the following Strichartz estimates (see [GV95, LS95]).

Proposition 2.1. *Let u and f be as above. Then $u \in C^0(0, T; \dot{H}^1)$ and $\partial_t u \in C^0(0, T; L^2)$. Furthermore, for a constant $C > 0$ independent of $T \in [0, \infty]$*

$$(2.5) \quad \|u\|_{\ell(0, T)} + \sup_{t \in [0, T]} \|\nabla u(t)\|_2 + \|\partial_t u(t)\|_2 \leq C \left(\|\nabla u_0\|_2 + \|u_1\|_2 + \|D_x^{1/2}f\|_{N(0, T)} \right).$$

Furthermore, if $D_x^{1/2}f \in N(T, +\infty)$,

$$(2.6) \quad \forall t \geq 0, \quad \left\| \nabla \left(\int_T^{+\infty} \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} f(s)ds \right) \right\|_2 + \left\| \partial_t \left(\int_T^{+\infty} \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} f(s)ds \right) \right\|_2 \leq C \|D_x^{1/2}f\|_{N(T, +\infty)}.$$

A solution of (1.1) on an interval $I \ni 0$ is a function $u \in C^0(I, \dot{H}^1)$ such that $\partial_t u \in C^0(I, \dot{H}^1)$ and $u \in S(J)$ for all interval $J \Subset I$ and

$$u(t, x) = \cos(t\sqrt{-\Delta})u_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}u_1 + \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}|u(s)|^{\frac{4}{N-2}}u(s)ds.$$

Proposition 2.2. (see [Pec84, GSV92, SS94])

- (a) Existence. *If $u_0 \in \dot{H}^1$, $u_1 \in L^2$, there exists an interval $I \ni 0$ and a solution u of (1.1) on I with initial conditions (u_0, u_1) .*

- (b) Uniqueness. *If u and \tilde{u} are solutions of (1.1) on an interval $I \ni 0$ such that $u(0) = \tilde{u}(0)$ and $\partial_t u(0) = \partial_t \tilde{u}(0)$, then $u = \tilde{u}$ on I .*

According to Proposition 2.2, if $(u_0, u_1) \in \dot{H}^1 \times L^2$, there exists a maximal open interval of definition for the solution u of (1.1), that we will denote by $(-T_-(u), T_+(u))$. The following holds

Proposition 2.3.

- (a) Finite blow-up criterion. *If $T_+ := T_+(u) < \infty$ then*

$$\|u\|_{S(0, T_+)} = \infty.$$

A similar result holds for negative times.

- (b) Continuity. *Let u be a solution of (1.1) on an interval I with initial condition $(u_0, u_1) \in \dot{H}^1 \times L^2$. If (u^k) is a sequence of solution of (1.1) with initial conditions*

$$(u_0^k, u_1^k) \xrightarrow{k \rightarrow +\infty} (u_0, u_1) \text{ in } \dot{H}^1 \times L^2$$

and $J \Subset (-T_-, T_+)$, then for large k , $J \subset (-T_-(u^k), T_+(u^k))$, and

$$(u^k, \partial_t u^k) \xrightarrow{k \rightarrow +\infty} (u, \partial_t u) \text{ in } C^0(J, \dot{H}^1) \times C^0(J, L^2), \quad u^k \xrightarrow{k \rightarrow +\infty} u \text{ in } S(J).$$

- (c) Scattering. *If u is a solution of (1.1) such that $T_+(u) = \infty$ and $\|u\|_{S(0, \infty)} < \infty$, then u scatters.*
- (d) Finite speed of propagation. *(see [KM06b, Lemma 2.17]) There exist $\varepsilon_0, C_0 > 0$, depending only on $\|\nabla u_0\|_2$ and $\|u_1\|_2$, such that if there exist $M, \varepsilon > 0$ satisfying $\varepsilon < \varepsilon_0$ and $\int_{|x| \geq M} |\nabla_x u_0| + \frac{1}{|x|^2} |u_0|^2 + |u_0|^{2^*} + |u_1|^2 \leq \varepsilon$, then,*

$$\forall t \in [0, T_+(u)), \quad \int_{|x| \geq \frac{3}{2}M+t} |\nabla u(t, x)|^2 + \frac{1}{|x|^2} |u(t, x)|^2 + |u|^{2^*} + |\partial_t u(t, x)|^2 dx \leq C_0 \varepsilon.$$

2.2. Properties of subcritical threshold solutions. We are now interested in solutions of (1.1) with maximal interval of definition (T_-, T_+) and such that

$$(2.7) \quad E(u_0, u_1) = E(W, 0), \quad \|\nabla u_0\|_2 < \|\nabla W\|_2$$

$$(2.8) \quad \|u\|_{S(0, T_+)} = \infty.$$

We start with the following claim (see [KM06b, Theorem 3.5]).

Claim 2.4 (Energy Trapping). *Let u be a solution of (1.1) satisfying (2.7). Then for all t in the interval of existence $(-T_-, T_+)$ of u .*

$$(2.9) \quad \|\nabla u(t)\|_2^2 + \frac{N}{2} \|\partial_t u\|_2^2 \leq \|\nabla W\|_2^2.$$

Proof. Recall the following property which follows from a convexity argument

$$(2.10) \quad \forall v \in \dot{H}^1, \quad \|\nabla v\|_2^2 \leq \|\nabla W\|_2^2 \text{ and } E(v, 0) \leq E(W, 0) \implies \frac{\|\nabla v\|_2^2}{\|\nabla W\|_2^2} \leq \frac{E(v, 0)}{E(W, 0)}.$$

(See [DM07, Claim 2.6]). Let u be as in the claim. Note that by remark 1.3 $\|\nabla u(t)\|_2 < \|\nabla W\|_2$ for all t in the domain of existence of u . Now, according to (2.10) and the fact that $E(u(t), \partial_t u(t)) = E(W, 0)$,

$$\frac{\|\nabla u(t)\|_2^2}{\|\nabla W\|_2^2} \leq \frac{E(u, \partial_t u) - \frac{1}{2} \|\partial_t u(t)\|_2^2}{E(W, 0)} = \frac{E(W, 0) - \frac{1}{2} \|\partial_t u(t)\|_2^2}{E(W, 0)},$$

which yields (2.9), recalling that $NE(W, 0) = \|\nabla W\|_2^2$. \square

We recall now some key results from [KM06b]. In their work, these results are shown for a critical element u , where assumptions (2.7) are replaced by $E(u_0, u_1) < E(W, 0)$, and $\|\nabla u_0\|_2 < \|\nabla W\|_2$. Rather than recalling the proofs which are long and far from being trivial, we will briefly explain how they adapt in our case. If (f, g) is in $\dot{H}^1 \times L^2$, we write

$$(f, g)_{\lambda_0, x_0}(y) = \left(\frac{1}{\lambda_0^{\frac{N}{2}-1}} f\left(\frac{y}{\lambda_0} + x_0\right), \frac{1}{\lambda_0^{\frac{N}{2}}} g\left(\frac{y}{\lambda_0} + x_0\right) \right), \quad f_{\lambda_0, x_0}(y) = \frac{1}{\lambda_0^{\frac{N}{2}-1}} f\left(\frac{y}{\lambda_0} + x_0\right).$$

Lemma 2.5. *Let u be a solution of (1.1) satisfying (2.7) and (2.8). Then there exist continuous functions of t , $(\lambda(t), x(t))$ such that*

$$K := \left\{ (u(t), \partial_t u(t))_{\lambda(t), x(t)}, t \in [0, T_+) \right\}$$

has compact closure in $\dot{H}^1 \times L^2$.

The proof of Lemma 2.5, which corresponds to Proposition 4.2 in [KM06b], is very close to the proof of Proposition 4.1 in [KM06a] and of Proposition 2.1 in [DM07]. The two main ingredients are the fact, proven in [KM06b] that a solution of (1.1) such that $E(u_0, u_1) < E(W, 0)$ and $\|\nabla u_0\|_2 < \|\nabla W\|_2$ is globally defined and scatters, a Lemma of concentration-compactness for solution to the linear wave equation due to Bahouri and Gérard [BG99], and variational estimates as in Claim 2.4.

Proposition 2.6 ([KM06b]). *Let u be a solution of (1.1) satisfying (2.7) and (2.8). Assume that there exist functions $(\lambda(t), x(t))$ such that*

$$K := \left\{ (u(t), \partial_t u(t))_{\lambda(t), x(t)}, t \in [0, T_+) \right\}$$

has compact closure in $\dots \times L^2$ and that one of the following holds

- (a) $T_+ < \infty$, or
- (b) $T_+ = +\infty$ and there exists $\lambda_0 > 0$ such that $\forall t \in [0, +\infty)$, $\lambda(t) \geq \lambda_0$.

Then $\int u_1 \nabla u_0 = 0$.

Proof. If $\inf_{t \in (-T_-, T_+)} \|\partial_t u(t)\|_2^2 = 0$, then, using that $\|\nabla u(t)\|_2$ is bounded, and that $\int \partial_t u \nabla u(t)$ is conserved, we get immediately that $\int u_1 \nabla u_0 = 0$. Thus we may assume

$$(2.11) \quad \exists \delta_0 > 0, \forall t \in (-T_-, T_+), \quad \|\nabla u(t)\|_2^2 \leq \|\nabla W\|_2^2 - \delta_0.$$

In this case, the proof is the same as in [KM06b, Propositions 4.10 and 4.11] which is shown under assumption (2.8) and

$$(2.12) \quad \|\nabla u_0\|_2 < \|\nabla W\|_2, \quad E(u_0, u_1) < E(W, 0).$$

This implies by variational estimates (2.11), which is what is really needed in the proof of the proposition. \square

Proposition 2.7 ([KM06b]). *Let u be a solution of (1.1) satisfying (2.7) and (2.8). Assume*

$$(2.13) \quad \int u_1 \nabla u_0 = 0.$$

Then $T_+ = \infty$.

This result is proven in [KM06b, Section 6] under the assumptions (2.8) and (2.12), but assumption (2.12) is only used to show that $\|\nabla u(t)\|_2$ is bounded, which is, in our case, a consequence of (2.7) (see [KM06b, Remark 6.14]).

As a consequence of Proposition 2.6 and 2.7 we have, following again [KM06b]:

Proposition 2.8. *Let u be a solution of (1.1) satisfying (2.7) and (2.8). Let $\lambda(t)$, $x(t)$ given by Lemma 2.5. Then*

- (a) $T_+ = \infty$.
- (b) $\lim_{t \rightarrow +\infty} t\lambda(t) = +\infty$.
- (c) $\int_{\mathbb{R}^N} u_1 \nabla u_0 = 0$.
- (d) $\lim_{t \rightarrow +\infty} \frac{x(t)}{t} = 0$.

Corollary 2.9. *Let u be a solution of (1.1) with maximal interval of definition $(-T_-, T_+)$ and such that $E(u_0, u_1) \leq E(W, 0)$ and $\|\nabla u_0\|_2 \leq \|\nabla W\|_2$. Then*

$$T_+ = T_- = +\infty.$$

Proof of Corollary 2.9. It is a consequence of (a). Indeed, if $\|\nabla u_0\|_2 = \|\nabla W\|_2$, then by Claim 2.4, $u_1 = 0$. Furthermore by (2.10), $E(W, 0) = E(u_0, 0) = E(u_0, u_1)$. Thus $\|u_0\|_{2^*} = \|W\|_{2^*}$, and, by the characterization (1.4) of W , $u_0 = \pm W_{\lambda_0, x_0}$ for some parameters λ_0, x_0 . By uniqueness in (1.1), u is one of the stationary solutions $\pm W_{\lambda_0, x_0}$, which are globally defined.

Let us assume now $\|\nabla u_0\|_2 < \|\nabla W\|_2$. Then if $E(u_0, u_1) < E(W, 0)$, we are in the setting of [KM06b, Theorem 1.1], which asserts that $T_+ = T_- = +\infty$. On the other hand, if $E(u_0, u_1) = E(W, 0)$, then if $\|u\|_{S(0, T_+)} < \infty$, we know by the finite blow-up criterion of Proposition 2.3, that $T_+ = \infty$, and if $\|u\|_{S(0, T_+)} = \infty$, then by (a), $T_+ = \infty$. The same argument for negative times shows that $T_- = \infty$. \square

Proof of Proposition 2.8. Proof of (a). Let u be as in Lemma 2.5. By Proposition 2.6, if $T_+ < \infty$, then $\int u_1 \nabla u_0 = 0$, but then by Proposition 2.7, $T_+ = \infty$ which is a contradiction. Thus $T_+ = \infty$, which shows (a).

Proof of (b). Assume that (b) does not hold. Then there exists a sequence $t_n \rightarrow +\infty$ such that

$$(2.14) \quad \lim_{n \rightarrow +\infty} t_n \lambda(t_n) = \tau_0 \in [0, +\infty).$$

Consider

$$(2.15) \quad w_n(s, y) = \frac{1}{\lambda(t_n)^{\frac{N-2}{2}}} u \left(t_n + \frac{s}{\lambda(t_n)}, x(t_n) + \frac{y}{\lambda(t_n)} \right).$$

$$(2.16) \quad w_{n0}(y) = w_n(0, y), \quad w_{n1}(y) = \partial_s w_n(0, y).$$

By the compactness of K , $(w_{n0}, w_{n1})_n$ converges (up to the extraction of a subsequence) in $\dot{H}^1 \times L^2$. Let (w_0, w_1) be its limit, and w be the solution of (1.1) with initial condition (w_0, w_1) . Note that $E(w_0, w_1) = E(W, 0)$ and $\|\nabla w_0\|_2 \leq \|\nabla W\|_2$. Thus by Corollary 2.9

$$T_-(w_0, w_1) = \infty.$$

Furthermore, as $-t_n\lambda(t_n) \rightarrow -\tau_0$, and by the continuity of the Cauchy Problem for (1.1) ((b) in Proposition 2.3),

$$\begin{aligned} \frac{1}{\lambda(t_n)^{\frac{N-2}{2}}} u_0 \left(x(t_n) + \frac{y}{\lambda(t_n)} \right) &= w_n(-t_n\lambda(t_n), y) \xrightarrow{n \rightarrow \infty} w(-\tau_0, y) \text{ in } \dot{H}^1 \\ \frac{1}{\lambda(t_n)^{\frac{N}{2}}} u_1 \left(x(t_n) + \frac{y}{\lambda(t_n)} \right) &= \partial_s w_n(-t_n\lambda(t_n), y) \xrightarrow{n \rightarrow \infty} \partial_s w(-\tau_0) \text{ in } L^2. \end{aligned}$$

Since $\lambda(t_n)$ tends to 0, we obtain that $w(-\tau_0) = 0$ and $\partial_s w(-\tau_0) = 0$, which contradicts the equality $E(w_0, w_1) = E(W, 0)$. The proof of (b) is complete.

Proof of (c). According to (a), $T_+ = \infty$. By Proposition 2.7, (c) holds unless

$$(2.17) \quad \liminf_{t \geq 0} \lambda(t) = 0.$$

Let us show (c) in this case. We will use the same argument as in the proof of Theorem 7.1 in [KM06b]. Let us sketch it. Consider $(t_n)_n$ such that

$$(2.18) \quad t_n \xrightarrow{n \rightarrow \infty} +\infty, \quad \lambda(t_n) \xrightarrow{n \rightarrow \infty} 0, \quad \text{and} \quad \forall t \in [0, t_n), \quad \lambda(t) > \lambda(t_n).$$

Define w_n , w_{n0} and w_{n1} by (2.15) and (2.16), and consider $(w_0, w_1) \in \dot{H}^1 \times L^2$ such that

$$(2.19) \quad \lim_{n \rightarrow +\infty} (w_{n0}, w_{n1})_n = (w_0, w_1) \text{ in } \dot{H}^1 \times L^2,$$

and w the solution of (1.1) with initial conditions (w_0, w_1) .

By Corollary 2.9, $T_-(w) = \infty$. By (2.8) and (b) in Proposition 2.3,

$$(2.20) \quad \|w\|_{S(-\infty, 0)} = +\infty.$$

Now, fix $s \leq 0$, and consider

$$\tilde{\lambda}_n(s) = \frac{\lambda\left(t_n + \frac{s}{\lambda(t_n)}\right)}{\lambda(t_n)}, \quad \tilde{x}_n(s) = \lambda(t_n) \left[x\left(t_n + \frac{s}{\lambda(t_n)}\right) - x(t_n) \right].$$

By (b), $t_n\lambda(t_n) \rightarrow +\infty$, and thus for large n , $0 < t_n + \frac{s}{\lambda(t_n)} \leq t_n$. Hence by (2.18),

$$(2.21) \quad \exists n_0(s), \forall n \geq n_0(s), \quad \tilde{\lambda}_n(s) \geq 1.$$

Now, for $t = t(n, s) := t_n + \frac{s}{\lambda_n(s)}$,

$$(2.22) \quad v_{n0}(s, y) := \frac{1}{\tilde{\lambda}_n(s)^{\frac{N-2}{2}}} w_n \left(s, \tilde{x}_n(s) + \frac{y}{\tilde{\lambda}_n(s)} \right) = \frac{1}{\lambda(t)^{\frac{N-2}{2}}} u \left(t, x(t) + \frac{y}{\lambda(t)} \right),$$

$$(2.23) \quad v_{n1}(s, y) := \frac{1}{\tilde{\lambda}_n(s)^{\frac{N}{2}}} (\partial_s w_n) \left(s, \tilde{x}_n(s) + \frac{y}{\tilde{\lambda}_n(s)} \right) = \frac{1}{\lambda(t)^{\frac{N}{2}}} (\partial_t u) \left(t, x(t) + \frac{y}{\lambda(t)} \right).$$

which shows that $(v_{n0}(s), v_{n1}(s)) \in K$. In view of (2.19) and the continuity property (b) in Proposition 2.3, $(w_n(s), \partial_s w_n(s))$ converges in $\dot{H}^1 \times L^2$. By the compactness of \bar{K} and (2.21) this shows that there exists $\tilde{\lambda}(s) \in [1, +\infty)$, $\tilde{x}(s) \in \mathbb{R}^N$ such that for some subsequences,

$$\lim_{n \rightarrow +\infty} \tilde{\lambda}_n(s) = \tilde{\lambda}(s), \quad \lim_{n \rightarrow +\infty} \tilde{x}_n(s) = \tilde{x}(s)$$

and

$$\left(\frac{1}{\tilde{\lambda}(s)^{\frac{N-2}{2}}} w \left(s, \tilde{x}(s) + \frac{\cdot}{\tilde{\lambda}(s)} \right), \frac{1}{\tilde{\lambda}(s)^{\frac{N}{2}}} (\partial_s w) \left(s, \tilde{x}(s) + \frac{\cdot}{\tilde{\lambda}(s)} \right) \right) \in \overline{K}.$$

Thus w fullfills all the assumptions of Proposition 2.6, case (b), which shows that $\int w_1 \nabla w_0 = 0$. By (2.19) and the conservation of $\int \partial_t u(t) \nabla u(t)$

$$\int u_1 \nabla u_0 = \int \partial_t u(t_n) \nabla u(t_n) = \int w_{n1} \nabla w_{n0} \xrightarrow{n \rightarrow \infty} \int w_1 \nabla w_0 = 0.$$

The proof of (c) is complete.

Proof of (d).

We follow the lines of the proof of Lemma 5.5 in [KM06b]. Assume that (d) does not hold, and consider $t_n \rightarrow +\infty$ such that

$$\frac{|x(t_n)|}{t_n} \geq \varepsilon_0 > 0.$$

In particular, $x(t)$ is not bounded. We may assume that $x(0) = 0$, $\lambda(0) = 1$ and that x and λ are continuous. For $R > 0$, let

$$t_0(R) := \inf \{t \geq 0, |x(t)| \geq R\} \in [0, +\infty).$$

Thus $t_0(R)$ is well defined, $t_0(R) > 0$, $|x(t)| < R$ for $0 \leq t < t_0(R)$ and $|x(t_0(R))| = R$. As a consequence, if $R_n := |x(t_n)|$, then $t_n \geq t_0(R_n)$, which implies

$$(2.24) \quad \frac{R_n}{t_0(R_n)} \geq \varepsilon_0.$$

Let

$$(2.25) \quad e(u) := \frac{1}{2} |\partial_t u|^2 + \frac{1}{2} |\nabla u|^2 - \frac{1}{2^*} |u|^{2^*}, \quad r(u) := |\partial_t u|^2 + |\nabla u|^2 + \frac{1}{|x|^2} |u|^2 + |u|^{2^*}.$$

By compactness of K , we know that for each $\varepsilon > 0$, there exists $R_0(\varepsilon)$ such that

$$(2.26) \quad \forall t \geq 0, \quad \int_{\lambda(t)|x-x(t)| \geq R_0(\varepsilon)} r(u) dx \leq \varepsilon.$$

Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be radial, nonincreasing and such that $\varphi(x) = 1$ for $|x| \leq 1$ and $\varphi(x) = 0$ if $|x| \geq 2$. Let $\psi_R(x) := x\varphi(\frac{x}{R})$. Let $\varepsilon > 0$, to be chosen later independently of n and

$$\begin{aligned} \tilde{R}_n &:= \frac{R_0(\varepsilon)}{\inf_{t \in [0, t_0(R_n)]} \lambda(t)} + |x(t_n)| = \frac{R_0(\varepsilon)}{\inf_{t \in [0, t_0(R_n)]} \lambda(t)} + R_n \\ z_{\tilde{R}_n}(t) &:= \int_{\mathbb{R}^N} \psi_{\tilde{R}_n}(x) e(u)(t, x) dx. \end{aligned}$$

Note that

$$(2.27) \quad 0 \leq t \leq t_0(R_n), |x| \geq \tilde{R}_n \implies \lambda(t)|x - x(t)| \geq \lambda(t)(\tilde{R}_n - R_n) \geq R_0(\varepsilon).$$

Step 1. Bound on $z'_{\tilde{R}_n}(t)$. Let us show

$$(2.28) \quad \exists C_1 > 0, \forall n, \quad 0 \leq t \leq t_0(R_n) \implies |z'_{\tilde{R}_n}(t)| \leq C_1 \varepsilon.$$

Indeed by explicit calculation and equation (1.1) (see [KM06b, Lemma 5.3]), we get recalling that by (c) and the conservation of the moment $\int \partial_t u(t) \nabla u(t) = \int u_1 \nabla u_0 = 0$

$$(2.29) \quad z'_{\tilde{R}_n}(t) = \int_{\mathbb{R}^N} \partial_t u \nabla u + O\left(\int_{|x| \geq \tilde{R}_n} r(u) dx\right) = O\left(\int_{|x| \geq \tilde{R}_n} r(u) dx\right)$$

Estimate (2.28) then follows from (2.26) and (2.27)

Step 2. Bounds on $z_{\tilde{R}_n}(0)$ and $z_{\tilde{R}_n}(t_0(R_n))$. We next show

$$(2.30) \quad |z_{\tilde{R}_n}(0)| \leq 2\varepsilon \tilde{R}_n + MR_0(\varepsilon)$$

$$(2.31) \quad R_n(E(W) - \varepsilon) - 2\tilde{R}_n\varepsilon - M \frac{R_0(\varepsilon)}{\lambda(t_0(R_n))} \leq |z_{\tilde{R}_n}(t_0(R_n))|,$$

where $M := \sup_{t \geq 0} \int r(u)(t, x) dx \leq C \|\nabla W\|_2^2$ by Claim 2.4. We have.

$$|z_{\tilde{R}_n}(0)| = \int_{|x| \geq R_0(\varepsilon)} \psi_{\tilde{R}_n} e(u) dx + \int_{|x| \leq R_0(\varepsilon)} \psi_{\tilde{R}_n} e(u) dx.$$

Recall that $|\psi_{\tilde{R}_n}(x)| \leq |x| \leq 2\tilde{R}_n$. According to (2.26) (using that $x(0) = 0$ and $\lambda(0) = 1$), the first term is bounded by $2\tilde{R}_n\varepsilon$. The second term is bounded by $R_0(\varepsilon) \int r(u) dx$, which yields (2.30).

Write, for $t \in [0, t_0(R_n)]$

$$(2.32) \quad z_{\tilde{R}_n}(t) = \int_{\lambda(t)|x-x(t)| \geq R_0(\varepsilon)} \psi_{\tilde{R}_n} e(u) dx + \int_{\lambda(t)|x-x(t)| \leq R_0(\varepsilon)} \psi_{\tilde{R}_n} e(u) dx$$

Using again (2.26), we get $\left| \int_{\lambda(t)|x-x(t)| \geq R_0(\varepsilon)} \psi_{\tilde{R}_n} e(u) dx \right| \leq 2\tilde{R}_n\varepsilon$. According to (2.27) and the definition of $\psi_{\tilde{R}_n}$, if $\lambda(t)|x-x(t)| < R_0(\varepsilon)$, then $|x| < \tilde{R}_n$ which implies $\psi_{\tilde{R}_n}(x) = x$. Thus the second term of (2.32) is

$$(2.33) \quad \begin{aligned} & \int_{\lambda(t)|x-x(t)| \leq R_0(\varepsilon)} x e(u) dx \\ &= \int_{\lambda(t)|x-x(t)| \leq R_0(\varepsilon)} x(t) e(u) dx + \int_{\lambda(t)|x-x(t)| \leq R_0(\varepsilon)} (x - x(t)) e(u) dx. \end{aligned}$$

The second term in the right-hand side of (2.33) is bounded by $\frac{R_0(\varepsilon)}{\lambda(t)} \int r(u)(t, x) dx$. On the other hand

$$\int_{\lambda(t)|x-x(t)| \leq R_0(\varepsilon)} x(t) e(u) dx = x(t) E(W) - \int_{\lambda(t)|x-x(t)| \geq R_0(\varepsilon)} x(t) e(u) dx$$

and thus by (2.26)

$$(2.34) \quad \left| \int_{\lambda(t)|x-x(t)| \leq R_0(\varepsilon)} x(t) e(u) dx \right| \geq |x(t)| (E(W) - \varepsilon).$$

Combining (2.33) and (2.34) with $t = t_0(R_n)$ we get (2.31).

Step 3. Conclusion of the proof of (d). According to the two precedent steps

$$(2.35) \quad \begin{aligned} C_1 \varepsilon t_0(R_n) &\geq R_n(E(W) - \varepsilon) - 4\tilde{R}_n \varepsilon - \frac{2MR_0(\varepsilon)}{\lambda(t_0(R_n))} \\ C_1 \varepsilon &\geq \frac{R_n}{t_0(R_n)}(E(W) - \varepsilon) - 4\frac{\tilde{R}_n}{t_0(R_n)}\varepsilon - M\frac{R_0(\varepsilon)}{t_0(R_n)\lambda(t_0(R_n))}. \end{aligned}$$

As a consequence of (b)

$$t_0(R_n)\lambda(t_0(R_n)) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Furthermore $\frac{\tilde{R}_n}{t_0(R_n)} = \frac{R_n}{t_0(R_n)} + \frac{R_0(\varepsilon)}{t_0(R_n)\inf_{t \in [0, t_0(R_n)]} \lambda(t)}$. Again by (b), $t_0(R_n) \inf_{t \in [0, t_0(R_n)]} \lambda(t)$ tends to $+\infty$. Thus

$$\frac{\tilde{R}_n}{t_0(R_n)} = \frac{R_n}{t_0(R_n)} + o(1), \quad n \rightarrow \infty.$$

Together with (2.24) and (2.35), we get

$$C_1 \varepsilon \geq \frac{R_n}{t_0(R_n)}(E(W) - 5\varepsilon) + o(1) \geq \varepsilon_0(E(W) - 5\varepsilon) + o(1), \quad n \rightarrow \infty.$$

Choosing ε small enough, so that $C_1 \varepsilon \leq \frac{\varepsilon_0}{4}E(W)$ and $E(W) - 5\varepsilon > \frac{1}{2}E(W)$ we get a contradiction. \square

3. CONVERGENCE TO W IN THE SUBCRITICAL CASE

The aim of this section is to prove the following result in the subcritical situation ($\|\nabla u_0\|_2 < \|\nabla W\|_2$), which is the nonradial version of the result of [DM07, Proposition 3.1] in the NLS radial setting. The main difficulty here compared to [DM07] and [KM06b] is to control the space localization of the energy (see §3.3).

Proposition 3.1. *Let u be a solution of (1.1) such that (2.7) and (2.8) hold. Then there exist λ_0, x_0 such that*

$$(3.1) \quad \|\nabla(u(t) - W_{\lambda_0, x_0})\|_2 + \|\partial_t u\|_2 \leq Ce^{-ct}.$$

Furthermore,

$$(3.2) \quad \|u\|_{S(-\infty, 0)} < \infty.$$

Remark 3.2. From Corollary 2.9, (2.7) implies that u is defined on \mathbb{R} .

3.1. Convergence for a subsequence. Let

$$(3.3) \quad d(t) := \left| \int |\nabla u(t, x)|^2 dx - \int |\nabla W(x)|^2 dx \right| + \int |\partial_t u(t, x)|^2 dx.$$

Then the equality $E(u(t), \partial_t u(t)) = E(W, 0)$ implies $\left| \|u\|_{2^*}^2 - \|W\|_{2^*}^2 \right| \leq Cd(t)$. It is known (see [Lio85]) that the variational characterization (1.4) of W by Aubin and Talenti implies that there exists a function $\varepsilon_0(\delta)$ such that $\varepsilon_0(\delta) \rightarrow 0$ as $\delta \rightarrow 0$ and, for any fixed t

$$(3.4) \quad \inf_{\mu, X, \pm} \|\nabla(u_{\mu, X}(t) \pm W)\|_2 \leq \varepsilon_0(d(t)).$$

The key point of the proof of Proposition 3.1 is to show that $d(t)$ tends to 0, which by (3.4) implies that there exists $(\lambda(t), x(t))_{t \geq 0}$ such that $u_{\lambda(t), x(t)}(t) - W$ tends to 0 in \dot{H}^1 as t tends to $+\infty$. We first show:

Lemma 3.3. *Let u be a solution of (1.1) such that (2.7) and (2.8) hold. Then*

$$(3.5) \quad \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \mathbf{d}(t) dt \xrightarrow{t \rightarrow +\infty} 0.$$

Corollary 3.4. *There exists an increasing sequence $\tau_n \rightarrow +\infty$ such that*

$$\lim_{n \rightarrow +\infty} \mathbf{d}(\tau_n) = 0.$$

Proof. Let φ be a C^∞ function such that $\varphi(x) = 1$ if $|x| \leq 1$ and $\varphi(x) = 0$ if $|x| \geq 2$. For $R > 0$, write

$$(3.6) \quad \varphi_R(x) = \varphi(x/R) \text{ and } \psi_R(x) = x\varphi(x/R).$$

Let $\varepsilon > 0$. Consider as in [KM06b, §5]

$$(3.7) \quad g_R(t) := \int \psi_R \cdot \nabla_x u \partial_t u dx + \left(\frac{N-1}{2}\right) \int \varphi_R u \partial_t u dx.$$

Step 1. Bound on $g_R(t)$. We first show

$$(3.8) \quad \exists C_1 > 0, \forall t \geq 0, \quad |g_R(t)| \leq C_1 R.$$

Indeed, note that $\text{supp } \varphi_R \cup \text{supp } \psi_R \subset \{|x| \leq 2R\}$, so that $|\psi_R(x)| \leq 2R$ and $|\varphi_R(x)| \leq \frac{2R}{|x|}$. Hence

$$|g_R(t)| \leq 2R \int |\nabla_x u \partial_t u| dx + \frac{N}{2} \int \frac{2R}{|x|} |u \partial_t u| dx,$$

which yields (3.8) by Hardy's inequality and Claim 2.4.

Step 2. Bound on $g'_R(t)$. There exists $C_2 > 0$, $c > 0$, such that for all $\varepsilon > 0$, there exists $t_1 = t_1(\varepsilon) > 0$ such that

$$(3.9) \quad \forall t \in [t_1, T], \quad g'_{\varepsilon T}(t) \leq -cd(t) + C_2 \varepsilon.$$

By explicit computation and the equality $E(u_0, u_1) = E(W, 0)$ (see Claim C.1 in the appendix)

$$g'_{\varepsilon T}(t) = \frac{1}{N-2} \int |\partial_t u|^2 dx - \frac{1}{N-2} \left(\int |\nabla W|^2 dx - \int |\nabla u|^2 dx \right) + O \left(\int_{|x| \geq \varepsilon T} r(u) dx \right),$$

where $r(u)$ is defined in (2.25). Note that by Claim 2.4 an elementary calculation

$$(3.10) \quad \begin{aligned} & -\frac{1}{N-2} \int |\partial_t u|^2 dx + \frac{1}{N-2} \left(\int |\nabla W|^2 dx - \int |\nabla u|^2 dx \right) \\ & \geq \frac{1}{N+2} \left(\int |\partial_t u|^2 dx + \int |\nabla W|^2 dx - \int |\nabla u|^2 dx \right). \end{aligned}$$

Thus there exists $C_2 > 0$ such that

$$(3.11) \quad g'_R(t) \leq -\frac{1}{N+2} \mathbf{d}(t) + C_2 \int_{|x| \geq \varepsilon T} r(u).$$

By Proposition 2.8, $t\lambda(t) \rightarrow +\infty$ and $|x(t)|/t \rightarrow 0$. Thus we may chose $t_1 = t_1(\varepsilon)$ such that

$$t \geq t_1 \implies |x(t)| \leq \frac{\varepsilon}{2} t \text{ and } |\lambda(t)| \geq \frac{2R_0(\varepsilon)}{\varepsilon t},$$

where $R_0(\varepsilon)$ is defined in (2.26) Then for $t_1 \leq t \leq T$,

$$|x| \geq \varepsilon T \implies \lambda(t)(|x| - |x(t)|) \geq \frac{2R_0(\varepsilon)}{\varepsilon T} \left(\varepsilon T - \frac{\varepsilon T}{2} \right) \geq R_0(\varepsilon),$$

which yields, together with (2.26) and (3.11), our expected estimate (3.9).

Step 3. End of the proof.

Integrating (3.9) between t_1 and T , one gets

$$g_{\varepsilon T}(T) - g_{\varepsilon T}(t_1) = \int_{t_1}^T g'_{\varepsilon T}(t) dt \leq -c \int_{t_1}^T d(t) dt + C_2(T - t_1)\varepsilon.$$

and thus, by (3.8),

$$\frac{c}{T} \int_{t_1}^T d(t) dt \leq 2C_1\varepsilon + C_2\varepsilon,$$

hence

$$\limsup_{T \rightarrow +\infty} \frac{1}{T} \int_0^T d(t) dt \leq \frac{2C_1 + C_2}{c} \varepsilon$$

which yields the result. \square

3.2. Modulation of solutions. We will now precise, using modulation theory, the dynamics of solutions of (1.1) near W . We will only suppose

$$(3.12) \quad E(u_0, u_1) = E(W, 0),$$

without any further assumption on the size of $\|\nabla u_0\|_2$. We have the following development of the energy near W :

$$(3.13) \quad E(W + f, g) = E(W, 0) + Q(f) + \frac{1}{2}\|g\|_2^2 + O(\|\nabla f\|_2^3), \quad f \in \dot{H}^1, g \in L^2$$

where Q is the quadratic form on \dot{H}^1 defined by

$$(3.14) \quad Q(f) := \frac{1}{2} \int |\nabla f|^2 - \frac{N+2}{2(N-2)} \int W^{\frac{4}{N-2}} f^2.$$

Let us specify an important coercivity property of Q . Consider the orthogonal directions \widetilde{W} , \widetilde{W}_j , $j = 1 \dots N$ in the real Hilbert space $\dot{H}^1 = \dot{H}^1(\mathbb{R}^N, \mathbb{C})$ where \widetilde{W} and \widetilde{W}_j are defined by

$$(3.15) \quad \begin{aligned} \widetilde{W} &= \tilde{c} \frac{\partial}{\partial \mu} (W_{\mu, X})|_{(\mu, X)=(1,0)} = -\tilde{c} \left(\frac{N-2}{2} W + x \cdot \nabla W \right) \\ \widetilde{W}_j &= c_j \frac{\partial}{\partial X_j} (W_{\mu, X})|_{(\mu, X)=(1,0)} = c_j \partial_{x_j} W. \end{aligned}$$

and the constants \tilde{c} , c_1, \dots, c_N are chosen so that

$$(3.16) \quad \|\nabla \widetilde{W}\|_2 = \|\nabla W_1\|_2 = \dots = \|\nabla W_N\|_2 = 1.$$

We have

$$(3.17) \quad Q(W) = -\frac{2}{(N-2)C_N^N}, \quad Q|_{\text{span}\{\widetilde{W}, W_1, \dots, W_N\}} = 0,$$

where C_N is the best Sobolev constant in dimension N . The first assertion follows from direct computation and the fact that $\|W\|_2^{2^*} = \|\nabla W\|_2^2 = \frac{1}{C_N^N}$. From (3.13), (3.15), and the invariance of E by all transformations $f \mapsto f_{\mu, X}$, we get that $Q(\widetilde{W}) = Q(W_1) = \dots = Q(W_N) = 0$.

Furthermore, it is easy to check that $\widetilde{W}, W_1, \dots, W_N$ are Q -orthogonal, which gives the second assertion.

Let $H := \text{span}\{W, \widetilde{W}, W_1, \dots, W_N\}$ and H^\perp its orthogonal subspace in the real Hilbert space \dot{H}^1 . The quadratic form Q is nonpositive on H . By the following claim, Q is positive definite on H^\perp (see [Rey90, Appendix D] for the proof).

Claim 3.5. *There is a constant $\tilde{c} > 0$ such that for all radial function \tilde{f} in H^\perp*

$$Q(\tilde{f}) \geq \tilde{c} \|\nabla \tilde{f}\|_2^2.$$

Now, let u be a solution of (1.1) satisfying (3.12) and define $d(t)$ as in (3.3). We would like to specify (3.4). We start by choosing μ and X .

Claim 3.6. *There exists $\delta_0 > 0$ such that for all solution u of (1.1) satisfying (3.12), and for all t in the interval of existence of u such that*

$$(3.18) \quad d(t) \leq \delta_0,$$

there exists $(\mu(t), X(t)) \in (0, +\infty) \times \mathbb{R}^N$ such that

$$u_{\mu, X} \in \{\widetilde{W}, W_1, \dots, W_N\}^\perp.$$

We omit the proof of Claim 3.6, which follows from (3.4) and the implicit function Theorem. We refer for example to [DM07, Claim 3.5] for a similar proof.

If a and b are positive, we write $a \approx b$ when $C^{-1}a \leq b \leq Ca$ with a positive constant C independent of all parameters of the problem.

Now, consider u satisfying (3.12) and assume that on an open subset J of its interval of definition, u also satisfies (3.18).

According to the preceding claim, there exist $(\mu(t), X(t))$ such that

$$\forall t \in J, \quad u_{\mu, X}(t) \in \{\widetilde{W}, W_1, \dots, W_N\}^\perp.$$

We will prove the following lemma, which is a consequence of Claim 3.5, in Appendix A.

Lemma 3.7 (Estimates on the modulation parameters). *Let u, μ, X be as above. Changing u into $-u$ if necessary, write*

$$u_{\mu, X}(t) = (1 + \alpha(t))W + \tilde{f}(t), \quad 1 + \alpha := \frac{1}{\|\nabla W\|_2^2} \int \nabla W \cdot \nabla u_{\mu, X} dx, \quad \tilde{f} \in H^\perp.$$

Then

$$(3.19) \quad |\alpha| \approx \|\nabla(\alpha W + \tilde{f})\| \approx \|\nabla \tilde{f}\|_2 + \|\partial_t u\|_2 \approx d(t)$$

$$(3.20) \quad \left| \frac{\alpha'}{\mu} \right| + \left| \frac{\mu'}{\mu^2} \right| + |X'(t)| \leq Cd(t).$$

3.3. Exponential convergence to W . Using Subsections 3.1 and 3.2, we are now ready to prove Proposition 3.1

Let u be as in the proposition. By Lemma 2.5, we may assume that

$$(3.21) \quad \text{There exist functions } \lambda(t), x(t) \text{ continuous on } [0, +\infty) \text{ such that} \\ K := \left\{ (u(t), \partial_t u(t))_{\lambda(t), x(t)}, t \in [0, +\infty) \right\} \text{ is relatively compact in } \dot{H}^1 \times L^2.$$

Let $\mu(t)$ and $X(t)$ be the modulation parameters of Subsection 3.2, defined for $d(t) \leq \delta_0$. It is easy to see that the compactness of K implies that the set

$$K_1 := \left\{ (u(t), \partial_t u(t))_{[\mu(t), X(t)]}, t \in [0, +\infty), d(t) < \delta_0 \right\}$$

has compact closure in $\dot{H}^1 \times L^2$. By an elementary construction, one can find continuous functions $\tilde{\lambda}(t)$ and $\tilde{x}(t)$ of $t \in [0, +\infty)$ such that $(\tilde{\lambda}(t), \tilde{x}(t)) = (X(t), \mu(t))$ if $d(t) \leq \delta_0$. The set \tilde{K} defined as in (3.21) has compact closure in $\dot{H}^1 \times L^2$. We will still denote by $x(t)$ and $\lambda(t)$ the new parameters that satisfy, in addition to (3.21),

$$(3.22) \quad d(t) < \delta_0 \implies \lambda(t) = \mu(t), \quad x(t) = X(t).$$

The proof of Proposition 3.1 relies on the two following Lemmas.

Lemma 3.8 (Virial type estimates on $d(t)$). *Let u be a solution of (1.1) satisfying (2.7), (3.21), and (3.22). Then there is a constant $C > 0$ such that*

$$0 \leq \sigma < \tau \implies \int_{\sigma}^{\tau} d(t) dt \leq C \left(\sup_{\sigma \leq t \leq \tau} |x(t)| + \frac{1}{\lambda(t)} \right) (d(\sigma) + d(\tau)).$$

Remark 3.9. We will also need the following variant of Lemma 3.8, whose proof is exactly the same: if u satisfies the assumptions of Lemma 3.8 for all $t \in \mathbb{R}$, then there is a constant $C > 0$

$$-\infty < \sigma < \tau < +\infty \implies \int_{\sigma}^{\tau} d(t) dt \leq C \left(\sup_{\sigma \leq t \leq \tau} |x(t)| + \frac{1}{\lambda(t)} \right) (d(\sigma) + d(\tau)).$$

Lemma 3.10 (Parameters control). *Let u be a solution of (1.1) fulfilling the assumptions of Lemma 3.8. Then there exists a constant $C > 0$ such that*

$$0 \leq \sigma \text{ and } \sigma + \frac{1}{\lambda(\sigma)} \leq \tau \implies |x(\tau) - x(\sigma)| + \left| \frac{1}{\lambda(\tau)} - \frac{1}{\lambda(\sigma)} \right| \leq C \int_{\sigma}^{\tau} d(t) dt.$$

Remark 3.11. The technical assumption $\sigma + \frac{1}{\lambda(\sigma)} < \tau$ is needed because of the infinite choice of parameters $x(t)$ and $\lambda(t)$ when $d(t) > \delta_0$.

Let us first assume Lemma 3.8 and 3.10 to show Proposition 3.1.

Proof of Proposition 3.1. Step 1. Let us show that, (replacing u by $u(\cdot - x_{\infty})$ for some $x_{\infty} \in \mathbb{R}^N$ if necessary), there exist $c, C > 0$ and $\lambda_{\infty} \in (0, \infty)$

$$(3.23) \quad \int_t^{\infty} d(s) ds + |\lambda(t) - \lambda_{\infty}| + |x(t)| \leq C e^{-ct}.$$

We first show that $x(t)$ and $\frac{1}{\lambda(t)}$ are bounded. According to Lemmas 3.8 and 3.10, there exists a constant $C_0 > 0$ such that for all $0 \leq \sigma \leq s < t \leq \tau$ with $s + \frac{1}{\lambda(s)} < t$, we have

$$(3.24) \quad |x(s) - x(t)| + \left| \frac{1}{\lambda(s)} - \frac{1}{\lambda(t)} \right| \leq C_0 \left\{ \sup_{\sigma \leq r \leq \tau} \left(|x(r)| + \frac{1}{\lambda(r)} \right) \right\} (d(\sigma) + d(\tau)).$$

Now consider the increasing sequence $\tau_n \rightarrow +\infty$, given by Corollary 3.4, and chose n_0 such that

$$(3.25) \quad n \geq n_0 \implies d(\tau_n) \leq \frac{1}{4C_0}$$

Using (3.24) with $\sigma = s = \tau_{n_0}$, and $\tau = \tau_n$ for some large n we get, in view of (3.25)

$$\tau_{n_0} + \frac{1}{\lambda(\tau_{n_0})} < t \implies |x(\tau_{n_0}) - x(t)| + \left| \frac{1}{\lambda(\tau_{n_0})} - \frac{1}{\lambda(t)} \right| \leq \frac{1}{2} \left\{ \sup_{\tau_{n_0} \leq r} \left(|x(r)| + \frac{1}{\lambda(r)} \right) \right\}.$$

Thus

$$\sup_{\tau_{n_0} + \frac{1}{\lambda(\tau_{n_0})} \leq t} \left(|x(t)| + \frac{1}{\lambda(t)} \right) \leq \frac{1}{2} \left\{ \sup_{\tau_{n_0} \leq t} \left(|x(t)| + \frac{1}{\lambda(t)} \right) \right\} + |x(\tau_{n_0})| + \frac{1}{\lambda(\tau_{n_0})},$$

which shows the boundedness of x and λ .

By Lemma 3.8 between $\sigma = t$ and $\tau = \tau_n$, and taking into account the fact that $x(t)$ and $\frac{1}{\lambda(t)}$ are bounded, we get $\int_t^{\tau_n} d(s)ds \leq C(d(t) + d(\tau_n))$. Letting n goes to infinity we obtain $\int_t^{+\infty} d(s)ds \leq Cd(t)$. Thus for some constants $c, C > 0$,

$$(3.26) \quad \int_t^{+\infty} d(s)ds \leq Ce^{-ct},$$

which is the first bound in (3.23).

By (3.26), Lemma 3.10 and the fact that $x(t)$ and $\frac{1}{\lambda(t)}$ are bounded, we have, if $\sigma + \frac{1}{\lambda(\sigma)} < \tau$, $|x(\sigma) - x(\tau)| + \left| \frac{1}{\lambda(\sigma)} - \frac{1}{\lambda(\tau)} \right| \leq Ce^{-c\sigma}$. Thus there exist $x_\infty \in \mathbb{R}^N$, $\ell_\infty \in [0, +\infty)$ such that

$$|x(t) - x_\infty| + \left| \frac{1}{\lambda(t)} - \ell \right| \leq Ce^{-ct}.$$

Translating u , we will assume $x_\infty = 0$. It remains to show that $\ell_\infty > 0$. Assume that $\ell_\infty = 0$. Let $0 \leq \sigma \leq s$. Let τ_n be the sequence such that $d(\tau_n) \rightarrow 0$. Then, by (3.24), if n is large enough (so that $s + \frac{1}{\lambda(s)} < \tau_n$),

$$|x(s) - x(\tau_n)| + \left| \frac{1}{\lambda(s)} - \frac{1}{\lambda(\tau_n)} \right| \leq C_0 \left[\sup_{\sigma \leq r \leq \tau_n} \left(|x(r)| + \frac{1}{\lambda(r)} \right) \right] (d(s) + d(\tau_n)).$$

Letting n tends to infinity, we get, by the assumptions $\ell_\infty = 0$, and $x_\infty = 0$

$$0 \leq \sigma \leq s \implies |x(s)| + \frac{1}{\lambda(s)} \leq C_0 \left[\sup_{\sigma \leq r} \left(|x(r)| + \frac{1}{\lambda(r)} \right) \right] d(\sigma).$$

Taking the supremum in s in the preceding inequality, we get, if $\sigma = \tau_n$

$$\sup_{\tau_n \leq s} |x(s)| + \frac{1}{\lambda(s)} \leq C_0 \left[\sup_{\tau_n \leq s} \left(|x(s)| + \frac{1}{\lambda(s)} \right) \right] d(\tau_n).$$

Recalling that $d(\tau_n)$ tends to 0, we get a contradiction, showing that $\ell_\infty > 0$. The proof of (3.23) is now complete.

Step 2. Proof of (3.1). Let us first show by contradiction

$$(3.27) \quad \lim_{t \rightarrow +\infty} d(t) = 0.$$

Indeed, if it does not hold, there exists a subsequence of $(\tau_n)_n$ (that we still denote by $(\tau_n)_n$), and a sequence $(\tilde{\tau}_n)_n$ such that

$$\tau_n < \tilde{\tau}_n, \quad \forall t \in [\tau_n, \tilde{\tau}_n), \quad d(t) < \delta_0/2, \quad d(\tilde{\tau}_n) = \delta_0/2.$$

On $[\tau_n, \tilde{\tau}_n]$ the parameters $\alpha(t)$, $\mu(t)$ and $X(t)$ of Lemma 3.7 are well-defined. By (3.23) and Lemma 3.7.

$$(3.28) \quad |\alpha(\tau_n) - \alpha(\tilde{\tau}_n)| \leq \int_{\tau_n}^{\tilde{\tau}_n} \left| \frac{\alpha'(t)}{\mu(t)} \right| dt \leq C \int_{\tau_n}^{\tilde{\tau}_n} d(t) dt \leq Ce^{-\tau_n}.$$

By Lemma 3.7, $\alpha(t) \approx d(t)$. As $d(\tau_n) \rightarrow 0$ and $d(\tilde{\tau}_n) = \delta_0/2$, this contradicts (3.28), showing (3.27).

In view of (3.27), there exists $T > 0$ such that for $t \geq T$, $d(t) < \delta_0$, so that u is close to $\pm W$ for $t \geq T$. By continuity of u , the sign before W does not change for large t . Changing u into $-u$ if necessary, we can make it a $+$. Write as in Lemma 3.7

$$(3.29) \quad u_{\mu, X}(t) = (1 + \alpha(t))W + \tilde{f}(t).$$

Integrating the estimate $|\alpha'(t)| \leq C\mu(t)d(t)$ of Lemma 3.7, we get, by (3.23), $|\alpha(t)| \leq Ce^{-ct}$. Furthermore, again by Lemma 3.7, $\|\nabla \tilde{f}\|_2 + \|\partial_t u\|_2 \lesssim d(t) \approx |\alpha(t)|$. Thus

$$(3.30) \quad \forall t \geq T, \quad |\alpha(t)| + |\mu(t) - \lambda_\infty| + |X(t)| + \|\nabla \tilde{f}\|_2 + \|\partial_t u(t)\|_2 \leq Ce^{-ct}.$$

This implies (3.1) in view of (3.29).

Step 3. Proof of (3.2). Assume, in addition to the assumption of Proposition 3.1, that we have

$$(3.31) \quad \|u\|_{S(-\infty, 0)} = +\infty.$$

By Lemma 2.5 there exist $\lambda(t)$ and $x(t)$, defined for $t \in \mathbb{R}$ such that

$$K := \left\{ (u(t), \partial_t u(t))_{\lambda(t), x(t)}, t \in \mathbb{R} \right\}$$

has compact closure in $\dot{H}^1 \times L^2$. As a consequence of the preceding steps, applied to $u(t, x)$ and $u(-t, x)$, we get that $d(t)$ tends to 0 as t goes to $+\infty$ and $-\infty$, and that $\frac{1}{\lambda(t)}$ and $x(t)$ are bounded independently of $t \in \mathbb{R}$. By Remark 3.9,

$$(3.32) \quad \sigma < \tau \Rightarrow \int_{\sigma}^{\tau} d(t) dt \leq C(d(\sigma) + d(\tau)).$$

Letting σ go to $-\infty$ and τ to $+\infty$, we get that $d(t) = 0$ for all t . Thus $u = W$ up to the invariance of the equation, which contradicts the assumption $\|\nabla u_0\|_2 < \|\nabla W\|_2$. \square

Proof of Lemma 3.8. Let $R > 0$ to be chosen later and g_R the function defined by (3.7)

Step 1. Bound on g_R . Let us show that there is a constant C_0 independent of $t \geq 0$ such that

$$(3.33) \quad |g_R(t)| \leq C_0 R d(t).$$

Indeed by the explicit expression of g_R , the fact that $\psi_R \leq 2R$ and $\varphi_R \leq 2R/|x|$ and Hardy's inequality we get

$$|g_R(t)| \leq CR \|\partial_t u(t)\|_2 \|\nabla u(t)\|_2 \leq CR \|\partial_t u(t)\|_2.$$

By Lemma 3.7 $\|\partial_t u\|_2 \leq Cd(t)$ for t such that $d(t) \leq \delta_0$. As $\|\partial_t u\|_2$ is bounded by $\sqrt{2E(W)}$ (Claim 2.4), this bounds is valid for any t , which concludes the proof of (3.33).

Step 2. Bound on g'_R . In this step we show that there exist $\rho_0 > 0$, $c > 0$, independent of σ and τ such that if for some $t \in [\sigma, \tau]$,

$$(3.34) \quad R \geq \rho_0 \left(\frac{1}{\lambda(t)} + |x(t)| \right),$$

then

$$(3.35) \quad g'_R(t) \leq -cd(t).$$

Indeed by Claim C.1 in the appendix,

$$g'_R(t) = \frac{1}{N-2} \int |\partial_t u|^2 dx - \frac{1}{N-2} \left(\int |\nabla W|^2 dx - \int |\nabla u|^2 dx \right) + A_R(u, \partial_t u),$$

where A_R is defined in (C.1). We first claim the following bounds on $A_R(u, \partial_t u)$:

$$(3.36) \quad \forall \varepsilon > 0, \exists \rho_\varepsilon > 0, \forall t \geq 0, \quad R \geq 2|x(t)| + \frac{\rho_\varepsilon}{\lambda(t)} \implies |A_R(u, \partial_t u)| \leq \varepsilon.$$

$$(3.37) \quad \exists C_1 > 0, \forall \rho \geq 1, \forall t \geq 0, \quad \left[R \geq 2|x(t)| + \frac{2\rho}{\lambda(t)} \text{ and } d(t) < \delta_0 \right] \\ \implies |A_R(u(t), \partial_t u(t))| \leq C_1 \left(\frac{1}{\rho^{\frac{N-2}{2}}} d(t) + d(t)^2 \right).$$

By (C.1), there exists $C_2 > 0$ such that

$$(3.38) \quad A_R(u, \partial_t u) \leq C_2 \int_{|x| \geq R} r(u) dx,$$

where $r(u)$ is defined in (2.25). Let $\rho_\varepsilon := 2R_0(\varepsilon/C_2)$, where R_0 is defined in (2.26). Assume that $R \geq 2|x(t)| + \frac{\rho_\varepsilon}{\lambda(t)}$. Then

$$|x| \geq R \implies |x - x(t)| \geq R - |x(t)| \geq \frac{R}{2} \geq \frac{R_0(\varepsilon/C_2)}{\lambda(t)}.$$

By (3.38) and the definition of R_0 , we get (3.36).

Let us show (3.37). Let t such that $d(t) < \delta_0$, where δ_0 is the parameter given by §3.2. Recall that by (3.22), $\lambda(t) = \mu(t)$ and $X(t) = x(t)$.

For any λ_0, x_0 , we know that W_{λ_0, x_0} is a solution of (1.1) independent of t , so that $g_R(t) = 0$, and $g'_R(t) = 0$, which shows by Claim C.1 that $A_R(W_{\lambda_0, x_0}, 0) = 0$. Thus

$$A_R(u, \partial_t u) = A_R(u, \partial_t u) - A_R\left(W_{\frac{1}{\mu}, -X}, 0\right).$$

By the change of variable $x = X + \frac{y}{\mu}$ we get

$$(3.39) \quad \int a_R^{jk}(x) \partial_j u(x) \partial_k u(x) dx - \int a_R^{jk} \frac{\partial}{\partial x_j} \left(W_{\frac{1}{\mu}, -X} \right) (x) \frac{\partial}{\partial x_k} \left(W_{\frac{1}{\mu}, -X} \right) (x) dx \\ = \int a_R^{jk} \left(X + \frac{y}{\mu} \right) \partial_j (W + f) \partial_k (W + f) dy - \int a_R^{jk} \left(X + \frac{y}{\mu} \right) \partial_j W \partial_k W dy \\ = \int a_R^{jk} \left(X + \frac{y}{\mu} \right) (\partial_j W \partial_k f + \partial_j f \partial_k W) dy + \int a_R^{jk} \left(X + \frac{y}{\mu} \right) \partial_j f \partial_k f dy.$$

where $f = u_{\mu,X} - W$, is such that $\|\nabla f(t)\|_2 \leq C_0 \mathbf{d}(t)$ by Lemma 3.7. Now, a similar calculation on the other terms of $A_R(u, \partial_t u) - A_R(W_{\frac{1}{\mu}, -X})$ yields the bound

$$(3.40) \quad |A_R(u, \partial_t u)| = \left| A_R(u, \partial_t u) - A_R\left(W_{\frac{1}{\mu}, -X}, 0\right) \right| \\ \leq C \int_{|X + \frac{y}{\mu}| \geq R} \left[|\nabla f|^2 + |\nabla W \cdot \nabla f| + W^{2^*-1}|f| + |f|^{2^*} + \frac{1}{\mu^2 |X + \frac{y}{\mu}|^2} (W|f| + |f|^2) \right] dy.$$

Let us bound the terms of the right-hand side of (3.40) that are linear in f . Recall that $\mu(t) = \lambda(t)$ and $X(t) = x(t)$. Using that $R \geq 2|x(t)| + \frac{2\rho}{\mu(t)}$, we get, if $|X + \frac{y}{\mu}| \geq R$

$$\frac{|y|}{\mu} \geq 2\frac{\rho}{\mu} \implies |y| \geq 2\rho \text{ and } \left| X + \frac{y}{\mu} \right| \geq \left| \frac{y}{\mu} \right| - X \geq \left| \frac{y}{\mu} \right| - \frac{R}{2} \geq \left| \frac{y}{\mu} \right| - \frac{\rho}{\mu} \geq \frac{1}{2} \left| \frac{y}{\mu} \right|.$$

Thus, recalling that $W(y) \approx |y|^{2-N}$ for large $|y|$.

$$\begin{aligned} \int_{|X + \frac{y}{\mu}| \geq R} |\nabla W \cdot \nabla f| dy &\leq \left(\int_{|y| \geq 2\rho} |\nabla W|^2 \right)^{1/2} \|\nabla f\|_2 \leq \frac{C}{\rho^{\frac{N-2}{2}}} \mathbf{d}(t) \\ \int_{|X + \frac{y}{\mu}| \geq R} W^{2^*-1} |f| dy &\leq \left(\int_{|y| \geq 2\rho} W^{2^*} \right)^{\frac{N+2}{2N}} \|f\|_{2^*} \leq \frac{C}{\rho^{\frac{N+2}{2}}} \mathbf{d}(t) \\ \int_{|X + \frac{y}{\mu}| \geq R} \frac{W|f|}{\mu^2 |X + \frac{y}{\mu}|^2} dy &\leq \int_{|X + \frac{y}{\mu}| \geq R} 4 \frac{W|f|}{|y|^2} \leq C \left(\int_{|y| \geq 2\rho} \frac{1}{|y|^2} |W|^2 \right)^{1/2} \|\nabla f\|_2 \leq \frac{C}{\rho^{\frac{N-2}{2}}} \mathbf{d}(t). \end{aligned}$$

By (3.40), we get (3.37).

We are now ready to show (3.35). Note that by Claim C.1, we have, for a small constant $\tilde{c} > 0$,

$$(3.41) \quad g'_R(t) \leq -\tilde{c} \mathbf{d}(t) + |A_R(u, \partial_t u)|.$$

Chose $\delta_1 := \min \left\{ \delta_0, \frac{\tilde{c}}{4C_1} \right\}$ and $\rho_1 > 1$ such that $\frac{C_1}{\rho_1^{\frac{N-2}{2}}} \leq \frac{\tilde{c}}{4}$ where C_1 is the constant in (3.37).

By (3.37)

$$\mathbf{d}(t) < \delta_1 \text{ and } R \geq 2 \left(|x(t)| + \frac{\rho_1}{\lambda(t)} \right) \implies |A_R(u, \partial_t u)| \leq \frac{\tilde{c}}{2} \mathbf{d}(t).$$

According to (3.36) with $\varepsilon := \frac{\tilde{c}}{2} \delta_1$,

$$\mathbf{d}(t) \geq \delta_1 \text{ and } R \geq 2|x(t)| + \frac{\rho_\varepsilon}{\lambda(t)} \implies |A_R(u, \partial_t u)| \leq \frac{\tilde{c}}{2} \delta_1 \leq \frac{\tilde{c}}{2} \mathbf{d}(t).$$

In view of (3.41), we get (3.35) under the assumption (3.34) for $\rho_0 := \max(2\rho_1, \rho_\varepsilon, 2)$. Step 2 is complete.

Step 3. End of the proof. Take

$$R := 2\rho_0 \sup_{\sigma \leq t \leq \tau} \left(\frac{1}{\lambda(t)} + |x(t)| \right),$$

where ρ_0 is given by Step 2. Then by (3.35)

$$\forall t \in [\sigma, \tau], \quad \mathbf{c} \mathbf{d}(t) \leq -g'_R(t)$$

Integrating between σ and τ , we get, in view of (3.33)

$$c \int_{\sigma}^{\tau} d(t) dt \leq |g_R(\sigma)| + |g_R(\tau)| \leq C_0 R(d(\sigma) + d(\tau)),$$

which yields the conclusion of Lemma 3.8. \square

Proof of Lemma 3.10. Step 1. Bounds by compactness on a short time interval. We show that there exists $C_1 > 0$ such that

$$(3.42) \quad \forall \tau, \sigma \geq 0, \quad |\tau - \sigma| \leq \frac{1}{\lambda(\tau)} \implies \lambda(\tau)|x(\tau) - x(\sigma)| + \frac{\lambda(\tau)}{\lambda(\sigma)} + \frac{\lambda(\sigma)}{\lambda(\tau)} \leq C_1.$$

If not, we may find sequences $\tau_n, \sigma_n \geq 0$ such that

$$(3.43) \quad |\tau_n - \sigma_n| \leq \frac{1}{\lambda(\tau_n)}, \quad \lambda(\tau_n)|x(\tau_n) - x(\sigma_n)| + \frac{\lambda(\tau_n)}{\lambda(\sigma_n)} + \frac{\lambda(\sigma_n)}{\lambda(\tau_n)} \xrightarrow{n \rightarrow +\infty} +\infty.$$

Extracting subsequences, we may assume

$$(3.44) \quad \lim_{n \rightarrow +\infty} \lambda(\tau_n)(\sigma_n - \tau_n) = s_0 \in [-1, 1].$$

Consider the solution of (1.1)

$$v_n(s, y) := \frac{1}{(\lambda(\tau_n))^{\frac{N-2}{2}}} u \left(\frac{s}{\lambda(\tau_n)} + \tau_n, \frac{y}{\lambda(\tau_n)} + x(\tau_n) \right).$$

By compactness of \overline{K} , extracting subsequences if necessary, $(v_n, \frac{\partial v_n}{\partial s})|_{s=0}$ has a limit (v_0, v_1) in $\dot{H}^1 \times L^2$. Let v be the solution of (1.1) with initial condition (v_0, v_1) , which is globally defined according to Corollary 2.9. By Proposition 2.3 (b),

$$w_n(y) := v_n(\lambda(\tau_n)(\sigma_n - \tau_n), y) = \frac{1}{(\lambda(\tau_n))^{\frac{N-2}{2}}} u \left(\sigma_n, \frac{y}{\lambda(\tau_n)} + x(\tau_n) \right) \xrightarrow{n \rightarrow +\infty} v(s_0, y) \text{ in } \dot{H}^1.$$

Furthermore the compactness of \overline{K} implies that the following sequence stays inside a compact set of \dot{H}^1 .

$$\frac{1}{(\lambda(\sigma_n))^{\frac{N-2}{2}}} u \left(\sigma_n, \frac{y}{\lambda(\sigma_n)} + x(\sigma_n) \right) = \left(\frac{\lambda(\tau_n)}{\lambda(\sigma_n)} \right)^{\frac{N-2}{2}} w_n \left(\frac{\lambda(\tau_n)}{\lambda(\sigma_n)} y + \lambda(\tau_n)(x(\sigma_n) - x(\tau_n)) \right).$$

Thus $\frac{\lambda(\tau_n)}{\lambda(\sigma_n)}$, $\frac{\lambda(\tau_n)}{\lambda(\sigma_n)}$ and $\lambda(\tau_n)(x(\tau_n) - x(\sigma_n))$ must be bounded, contradicting (3.43).

Step 2. Control of the variations of d . Let $\delta_0 > 0$ be given by Subsection 3.2. Let us show

$$(3.45) \quad \exists \delta_1 > 0, \quad \forall \tau \geq 0, \quad \sup_{\tau \leq t \leq \tau + \frac{1}{\lambda(\tau)}} d(t) > \delta_0 \implies \inf_{\tau \leq t \leq \tau + \frac{1}{\lambda(\tau)}} d(t) > \delta_1.$$

Indeed, assume that it does not hold, so that (extracting if necessary), we may find sequences $(\tau_n)_n, (t_n)_n, (t'_n)_n$, such that

$$(3.46) \quad t_n, t'_n \in \left[\tau_n, \tau_n + \frac{1}{\lambda(\tau_n)} \right], \quad d(t_n) \rightarrow 0 \text{ and } d(t'_n) > \delta_0.$$

Let

$$v_n(s, y) := \frac{1}{(\lambda(t_n))^{\frac{N-2}{2}}} u \left(\frac{s}{\lambda(t_n)} + t_n, \frac{y}{\lambda(t_n)} + x(t_n) \right).$$

By the compactness of K , and the fact that $d(t_n)$ tends to 0, we may assume that $(v_n(0), \partial_s v_n(0))$ tends to some W_{λ_0, x_0} .

By Step 1, $\frac{1}{\lambda(t_n)} \leq \frac{C_1}{\lambda(\tau_n)}$, thus $\lambda(t_n)(t_n - t'_n)$ is bounded. Extracting if necessary, we may assume $\lim_n \lambda(t_n)(t_n - t'_n) = s_0 \in [-1, 1]$. By Proposition 2.3 (b),

$$(3.47) \quad \lim_{n \rightarrow \infty} v_n(\lambda(t_n)(t_n - t'_n)) = \pm W_{\lambda_0, x_0} \text{ in } \dot{H}^1.$$

Furthermore, $v_n(\lambda(t_n)(t_n - t'_n)) = \frac{1}{\lambda(t_n)^{\frac{N-2}{2}}} u\left(t'_n, \frac{y}{\lambda(t_n)} + x(t_n)\right)$. Thus by (3.46), $\|\nabla W\|_2^2 - \|\nabla v_n\|_2^2 > \delta_0$, which contradicts (3.47). Step 2 is complete.

Step 3. End of the proof We first show that, there exists $C > 0$ such that

$$(3.48) \quad 0 \leq \sigma \leq \tilde{\sigma} \leq \tilde{\tau} \leq \tau = \sigma + \frac{1}{C_1 \lambda(\sigma)} \implies |x(\tilde{\tau}) - x(\tilde{\sigma})| + \left| \frac{1}{\lambda(\tilde{\tau})} - \frac{1}{\lambda(\tilde{\sigma})} \right| \leq C \int_{\sigma}^{\tau} d(r) dr,$$

where $C_1 \geq 1$ is the constant defined in Step 1. Indeed, if $d(t) \leq \delta_0$ for $t \in [\sigma, \tau]$, we have by (3.22) that $x(t) = X(t)$ and $\lambda(t) = \mu(t)$ on $[\sigma, \tau]$. Thus by (3.20) in Lemma 3.7,

$$|x(\tilde{\sigma}) - x(\tilde{\tau})| + \left| \frac{1}{\lambda(\tilde{\sigma})} - \frac{1}{\lambda(\tilde{\tau})} \right| = \left| \int_{\tilde{\sigma}}^{\tilde{\tau}} x'(t) dt \right| + \left| \int_{\tilde{\sigma}}^{\tilde{\tau}} \frac{\lambda'(t)}{\lambda^2(t)} dt \right| \leq C \int_{\sigma}^{\tau} d(t) dt,$$

which yields (3.48) in this case. The second case is when there exists a $t \in [\sigma, \tau]$ such that $d(t) > \delta_0$. By Step 2, we get that $d(t) > \delta_1$ for all $t \in [\sigma, \tau]$. Note that by Step 1, $|\tilde{\sigma} - \tilde{\tau}| \leq \frac{1}{C_1 \lambda(\sigma)} \leq \frac{1}{\lambda(\tilde{\sigma})}$, and thus, again by Step 1, $|x(\tilde{\sigma}) - x(\tilde{\tau})| \leq \frac{C_1}{\lambda(\tilde{\sigma})}$ and $\left| \frac{1}{\lambda(\tilde{\sigma})} - \frac{1}{\lambda(\tilde{\tau})} \right| = \frac{1}{\lambda(\tilde{\sigma})} \left| 1 - \frac{\lambda(\tilde{\sigma})}{\lambda(\tilde{\tau})} \right| \leq \frac{2C_1}{\lambda(\tilde{\sigma})}$.

$$|x(\tilde{\sigma}) - x(\tilde{\tau})| + \left| \frac{1}{\lambda(\tilde{\sigma})} - \frac{1}{\lambda(\tilde{\tau})} \right| \leq \frac{3C_1}{\lambda(\tilde{\sigma})} \leq \frac{3C_1^2}{\lambda(\sigma)} = 3C_1^2 |\sigma - \tau| \leq \frac{3C_1^2}{\delta_1} \int_{\sigma}^{\tau} d(t) dt.$$

The proof of (3.48) is complete.

It is straightforward to deduce the conclusion of Lemma 3.10 from (3.48), dividing the interval $[\sigma, \tau]$ into small subintervals, and we omit the details. \square

4. SUPERCRITICAL CASE FOR L^2 SOLUTIONS

In this section we study a solution u of (1.1) such that

$$(4.1) \quad u_0 \in L^2$$

$$(4.2) \quad E(u_0, u_1) = E(W, 0), \quad \|\nabla u_0\|_2 > \|\nabla W\|_2$$

$$(4.3) \quad T_+(u) = +\infty.$$

Our main result is the following.

Proposition 4.1. *Let u be a solution of (1.1) with $N \in \{3, 4, 5\}$ satisfying (4.1), (4.2) and (4.3). Then $N = 5$ and changing u into $-u$ if necessary, there exist $c, C > 0$ and λ_0, x_0 such that*

$$(4.4) \quad \forall t \geq 0, \quad \|\nabla u(t) - \nabla W_{\lambda_0, x_0}\|_2 + \|\partial_t u(t)\|_2 \leq C e^{-ct}.$$

Remark 4.2. In dimension $N = 3$ or $N = 4$, Proposition 4.1 asserts than any solution of (1.1) satisfying (4.1) and (4.2) must blow-up in finite time for positive and negative time. We are not able to prove (4.4). Nevertheless, one can show the weaker property

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T d(t) dt = 0.$$

Let

$$y(t) := \int_{\mathbb{R}^N} (u(t))^2 dx,$$

and define $d(t)$ by (3.3). We first prove the following.

Lemma 4.3. *Let u satisfying the assumptions of Proposition 4.1. Then*

$$(4.5) \quad \forall t \geq 0, \quad y'(t) < 0$$

$$(4.6) \quad \lim_{t \rightarrow +\infty} y(t) = y_\infty \in (0, +\infty)$$

$$(4.7) \quad \int_t^{+\infty} d(s) ds \leq C e^{-ct}.$$

Corollary 4.4. *Under the assumptions of Proposition 4.1, $T_-(u) < \infty$.*

Proof of Corollary 4.4. Indeed by (4.5), $y'(t) < 0$. But if $T_-(u) = +\infty$, (4.5) applied to the solution $u(-t, x)$ of (1.1) (which also satisfies the assumptions of Proposition 4.1) shows that $y'(t) > 0$, which is a contradiction. \square

Proof of Lemma 4.3. By direct calculation (and using equation (1.1) and assumption (4.2) to compute y'')

$$(4.8) \quad y'(t) = 2 \int_{\mathbb{R}^N} u(t) \partial_t u(t) dx$$

$$(4.9) \quad \begin{aligned} y''(t) &= 2 \int (\partial_t u(t))^2 - |\nabla u(t)|^2 + |u(t)|^{2^*} \\ &= 4 \frac{N-1}{N-2} \int (\partial_t u(t))^2 + \frac{4}{N-2} \left[\int |\nabla u(t)|^2 - \int |\nabla W|^2 \right] \geq d(t). \end{aligned}$$

Furthermore, by Cauchy-Schwarz inequality,

$$(4.10) \quad y'(t)^2 \leq 4 \int (u(t))^2 \int (\partial_t u(t))^2 \leq \frac{N-2}{N-1} y(t) y''(t).$$

Proof of (4.5). We argue by contradiction. Note that by Remark 1.3, assumption (4.2) implies that $\|\nabla u(t)\|_2 > \|\nabla W\|_2$ for all t . By (4.9), $y''(t) > 0$ for any $t \geq 0$. Assume that for some t_0 , $y'(t_0) \geq 0$.

$$(4.11) \quad \forall t > t_0, \quad y'(t) > 0.$$

Hence by (4.10), $\frac{N-1}{N-2} \frac{y'}{y} \leq \frac{y''}{y'}$, which yields by integration

$$\forall t \geq t_0 + 1, \quad \frac{y'(t)}{y'(t_0 + 1)} \geq \left(\frac{y(t)}{y(t_0 + 1)} \right)^{\frac{N-1}{N-2}},$$

which leads to blow-up in finite time from the fact that $\frac{N-1}{N-2} > 1$, contradicting (4.3).

Proof of (4.6). The function y is positive and, by (4.5), decreasing. Thus

$$(4.12) \quad \lim_{t \rightarrow +\infty} y(t) = y_\infty \in [0, +\infty).$$

We must show $y_\infty > 0$. Let us first show that for $t \geq 0$

$$(4.13) \quad |y'(t)| \leq C \|\partial_t u(t)\|_2 \leq C d(t).$$

By Cauchy-Schwarz, $|y'(t)| \leq \|u(t)\|_2 \|\partial_t u(t)\|_2$. By (4.12), $\|u(t)\|_2 = \sqrt{y(t)}$ is bounded, which shows the first bound in (4.13). According to Lemma 3.7, if $d(s) \leq \delta_0$ then $\|\partial_t u(t)\|_2 \leq C d(t)$. Furthermore, if $d(s) \geq \delta_0$, $\|\partial_t u(t)\|_2^2 \leq d(t) \leq \frac{1}{\delta_0} d(t)^2$, hence the bound $\|\partial_t u(t)\|_2 \leq C d(t)$, which concludes the proof of (4.13).

To show that $y_\infty > 0$, we argue by contradiction. Assume that $y_\infty = 0$. By (4.13),

$$(4.14) \quad y(t) = -(y_\infty - y(t)) = - \int_t^{+\infty} y'(s) ds \leq C \int_t^{+\infty} d(s) ds.$$

Note that

$$(4.15) \quad \int_t^{+\infty} d(s) ds \leq |y'(t)|.$$

Indeed $\int_t^T y''(s) ds = y'(T) - y'(t) \leq -y'(t)$, which yields (4.15) in view of (4.9). Combining (4.14) and (4.15), we get

$$\int_t^{+\infty} d(s) ds \leq |y'(t)| \leq 2 \|\partial_t u(t)\|_2 (y(t))^{1/2} \leq C \|\partial_t u(t)\|_2 \left(\int_t^{+\infty} d(s) ds \right)^{1/2}$$

and thus, by (4.13), $\left(\int_t^{+\infty} d(s) ds \right)^{1/2} \leq C d(t)$ for $t \geq 0$. This is a differential inequality of the form $\sqrt{Y} \leq -CY'$, which can not be valid on $[0, \infty)$ if $\forall t \geq 0, Y > 0$. The proof of (4.6) is complete.

Proof of (4.7). By (4.13) and (4.15),

$$\forall t \geq 0, \quad \int_t^\infty d(s) ds \leq |y'(t)| \leq C d(t),$$

which implies (4.7). □

Proof of Proposition 4.1. Step 1. Convergence in L^2 . We first show that there exists $u_\infty \in L^2$ such that

$$(4.16) \quad \lim_{t \rightarrow +\infty} \|u(t) - u_\infty\|_2 = 0.$$

Indeed we have, if $0 \leq t_1 < t_2$,

$$(4.17) \quad |u(t_1, x) - u(t_2, x)|^2 = \left| \int_{t_1}^{t_2} \partial_t u(t, x) dt \right|^2 \leq (t_2 - t_1) \int_{t_1}^{t_2} |\partial_t u(t, x)|^2 dt.$$

Integrating (4.17) in space, we get by (4.7)

$$\|u(t_1) - u(t_2)\|_2^2 \leq |t_1 - t_2| \int_{t_1}^{t_2} \|\partial_t u(t)\|_2^2 dt \leq C |t_1 - t_2| \int_{t_1}^{t_2} d(t) dt \leq C |t_1 - t_2| e^{-ct_1}.$$

By an elementary summation argument, we obtain, taking a larger constant C , the bound $\|u(t_1) - u(t_2)\|_2^2 \leq Ce^{-ct_1}$ for $t_1 < t_2$. Thus u satisfies the Cauchy criterion in L^2 as $t \rightarrow +\infty$, which yields (4.16).

Step 2. End of the proof By (4.7), there exists a sequence $t_n \rightarrow \infty$ such that $d(t_n)$ tends to 0. Thus, extracting a subsequence and changing u into $-u$ if necessary, there exists λ_0, x_0 such that $u(t_n)$ tends to W_{λ_0, x_0} in \dot{H}^1 , thus in $\mathcal{D}'(\mathbb{R}^N)$. In view (4.16), $u(t_n)$ tends also to u_∞ in $\mathcal{D}'(\mathbb{R}^n)$. Thus $W_{\lambda_0, x_0} = u_\infty \in L^2$. This shows that $N = 5$ and

$$(4.18) \quad \lim_{t \rightarrow +\infty} \|u(t) - W_{\lambda_0, x_0}\|_2 = 0.$$

Let us show

$$(4.19) \quad \lim_{t \rightarrow +\infty} d(t) = 0.$$

If (4.19) does not hold, there exist increasing sequences $(t_n)_n, (t'_n)_n$ such that $t_n < t'_n$, $d(t_n) \rightarrow 0$, $d(t'_n) = \delta_0$ and $d(t) < \delta_0$ for $t \in [t_n, t'_n]$. On $[t_n, t'_n]$, the modulation parameters $\mu(t)$ and $X(t)$ are well-defined. Furthermore, by (4.18), $\mu(t)$ must be bounded on $\bigcup_n [t_n, t'_n]$. Thus by (4.7) and the same argument as in the proof of (3.27), $\alpha(t'_n)$ tends to 0 which contradicts the estimate $d(t'_n) \approx \alpha(t'_n)$ of Lemma 3.7. Hence (4.19).

Thus there exists $T > 0$ such that for $t \geq T$, $d(t) \leq \delta_0$. By (4.18), μ converges to λ_0^{-1} . In view of estimate (3.20) of Lemma 3.7 and the boundedness of μ ,

$$(4.20) \quad |\alpha'(t)| + |\mu'(t)| + |X'(t)| \leq Cd(t).$$

In view of (4.7), this shows as in the end of the proof of Proposition 3.1 that $d(t)$, $\alpha(t)$, $\mu(t)$ and $X(t)$ converges exponentially when $t \rightarrow +\infty$, which implies (4.4). The proof of Proposition 4.1 is complete. \square

5. PRELIMINARIES ON THE LINEARIZED EQUATION NEAR W

This section is similar to the corresponding one in the NLS case [DM07, Section 5].

Let u be a solution of (1.1), defined on $[0, +\infty)$, and close to W . Let $h := u - W$. Then

$$\partial_t^2 h - \Delta h - |W + h|^{\frac{4}{N-2}}(W + h) + W^{\frac{N+2}{N-2}} = 0,$$

which we rewrite as

$$(5.1) \quad \begin{aligned} \partial_t^2 h + Lh &= R(h), \\ L &:= -\Delta - \frac{N+2}{N-2}W^{\frac{4}{N-2}}, \quad R(h) := |W + h|^{\frac{4}{N-2}}(W + h) - W^{\frac{N+2}{N-2}} - \frac{N+2}{N-2}W^{\frac{4}{N-2}}h. \end{aligned}$$

Note that $\frac{1}{2}(Lu, u)_{L^2} = Q(u)$ where Q is the quadratic form defined in (3.14).

5.1. Preliminary estimates. Recall the definition of the spaces $\ell(I)$ and $N(I)$ defined in (2.1), (2.2).

Lemma 5.1. *There exists $C > 0$ such that if $f \in L^{2*}$, I is a time interval and $u, v \in \ell(I)$.*

$$(5.2) \quad \|D_x^{1/2}(W^{\frac{4}{N-2}}u)\|_{N(I)} \leq C \left(|I|^{\frac{2}{N+1}} + |I|^{\frac{5}{2(N+1)}} \right) \|u\|_{\ell(I)}$$

$$(5.3) \quad \|R(f)\|_{\frac{2N}{N+2}} \leq C \left(\|f\|_{2^*}^2 + \|f\|_{2^*}^{\frac{N+2}{2}} \right)$$

$$(5.4) \quad \|D_x^{1/2}(R(u) - R(v))\|_{N(I)} \leq C \left(1 + |I|^{\frac{6-N}{2(N+1)}} \right) \|u - v\|_{\ell(I)} \left(\|u\|_{\ell(I)} + \|v\|_{\ell(I)} + \|u\|_{\ell(I)}^{\frac{4}{N-2}} + \|v\|_{\ell(I)}^{\frac{4}{N-2}} \right).$$

We postpone the proof of Lemma 5.1 to Appendix B.

We will also need the following version of Lemma 5.1 with exponentially decreasing norms.

Corollary 5.2. *Let $u, v \in \ell(t_0, +\infty)$, $t_0 \in \mathbb{R}$, such that for some $\gamma > 0$, and some constant $M > 0$,*

$$\forall t \geq t_0, \quad \|u\|_{\ell(t, +\infty)} + \|v\|_{\ell(t, +\infty)} \leq M e^{-\gamma t}.$$

Then there exists $C = C(\gamma, M) > 0$ such that

$$\|D_x^{1/2}(W^{\frac{4}{N-2}}u)\|_{N(t, +\infty)} \leq C e^{-\gamma t}, \quad \|R(u(t))\|_{\frac{2N}{N+2}} \leq C e^{-2\gamma t}$$

$$\forall t \geq t_0, \quad \|D_x^{1/2}(R(u) - R(v))\|_{N(t, +\infty)} \leq C e^{-\gamma t} \|u - v\|_{\ell(t, +\infty)}.$$

Proof. This is an immediate consequence of Lemma 5.1 and the following elementary Claim, which is Claim 5.8 in [DM07]:

Claim 5.3 (Sums of exponential). *Let $t_0 > 0$, $p \in [1, +\infty[$, $a_0 \neq 0$, E a normed vector space, and $f \in L_{\text{loc}}^p(t_0, +\infty; E)$ such that*

$$(5.5) \quad \exists \tau_0 > 0, \exists C_0 > 0, \forall t \geq t_0, \quad \|f\|_{L^p(t, t+\tau_0, E)} \leq C_0 e^{a_0 t}.$$

Then for $t \geq t_0$,

$$(5.6) \quad \|f\|_{L^p(t, +\infty, E)} \leq \frac{C_0 e^{a_0 t}}{1 - e^{a_0 \tau_0}} \text{ if } a_0 < 0; \quad \|f\|_{L^p(t_0, t, E)} \leq \frac{C_0 e^{a_0 t}}{1 - e^{-a_0 \tau_0}} \text{ if } a_0 > 0.$$

□

Corollary 5.4 (Strichartz estimates for the perturbative equation). *Let h be a solution of (5.1) on $[0, \infty)$ such that*

$$\|\nabla h(t)\|_2 + \|\partial_t h(t)\|_2 \leq C e^{-\gamma t}.$$

Then

$$\begin{aligned} \|h\|_{\ell(t, +\infty)} + \|D_x^{1/2} W^{\frac{4}{N-2}} h\|_{N(t, +\infty)} &\leq C e^{-\gamma t}, \\ \|R(h(t))\|_{\frac{2N}{N+2}} + \|D_x^{1/2} R(h)\|_{N(t, +\infty)} &\leq C e^{-2\gamma t}. \end{aligned}$$

Proof. The proof is the same than the one of [DM07, Lemma 5.6]. We sketch it for the sake of completeness. Note that all desired estimates are, by Corollary 5.2, a consequence of

$$\|h\|_{\ell(t, +\infty)} \leq C e^{-\gamma t},$$

so that we only need to show this last estimate. We have

$$\partial_t^2 h - \Delta h = \frac{N+2}{N-2} W^{\frac{4}{N-2}} h + R(h).$$

Let $t > 0$ and $\tau \in (0, 1)$. First note that $W + h$ is solution of (1.1), and thus, by the standard Cauchy problem theory for (1.1), $\|h\|_{\ell(t, t+\tau)}$ is finite. By Strichartz inequality (Proposition 2.1) and Lemma 5.1,

$$\begin{aligned} \|h\|_{\ell(t, t+\tau)} &\leq C \left(\|D_x^{1/2} W^{\frac{4}{N-2}} h\|_{N(t, t+\tau)} + \|D_x^{1/2} R(h)\|_{N(t, t+\tau)} + e^{-\gamma t} \right) \\ &\leq C \left(\tau^{\frac{2}{N+1}} \|h\|_{\ell(t, t+\tau)} + \|h\|_{\ell(t, t+\tau)}^2 + \|h\|_{\ell(t, t+\tau)}^{\frac{N+2}{N-2}} + e^{-\gamma t} \right). \end{aligned}$$

By a standard argument (see the proof of [DM07, Lemma 5.7]), we deduce from the preceding inequality, τ is small,

$$\|h\|_{\ell(t, t+\tau)} \leq C e^{-\gamma t}.$$

The conclusion follows from Claim 5.3. \square

5.2. Spectral theory for the linearized operator. The following Proposition sums up spectral properties of L (see [SK05], [SKT07] for the radial case in \mathbb{R}^3).

Proposition 5.5. *The operator L on L^2 with domain H^2 is a self-adjoint operator with essential spectrum $[0, +\infty)$, no positive eigenvalue and only one negative eigenvalue $-e_0^2$, with a radial, exponentially decreasing, smooth eigenfunction \mathcal{Y} . Furthermore, if*

$$G_{\perp} := \left\{ f \in \dot{H}^1, \quad \int \mathcal{Y} f = \int \nabla f \cdot \nabla \widetilde{W} = \int \nabla f \cdot \nabla W_1 = \dots = \int \nabla f \cdot \nabla W_N = 0 \right\}.$$

Then there exists $c_Q > 0$ such that

$$(5.7) \quad \forall f \in G_{\perp}, \quad Q(f) \geq c_Q \|\nabla f\|_2^2.$$

Remark 5.6. The proposition shows

$$\{u \in \dot{H}^1, Lu = 0\} = \text{span}\{\widetilde{W}, W_1, \dots, W_N\}.$$

Indeed, the inclusion \supset is already known. For the other inclusion, note that if the dimension of the space $\{u \in \dot{H}^1, Lu = 0\}$ was strictly higher than $N + 1$, we could find $\mathcal{Z} \in \dot{H}^1$, $\mathcal{Z} \neq 0$, such that $L\mathcal{Z} = 0$, and orthogonal to $\widetilde{W}, W_1, \dots, W_N$. By self-adjointness of L , $\int \mathcal{Z} \mathcal{Y} = -\frac{1}{e_0} \int \mathcal{Z} L\mathcal{Y} = 0$. Thus on one hand $\mathcal{Z} \in G_{\perp} \setminus \{0\}$, and on the other $Q(\mathcal{Z}) = (L\mathcal{Z}, \mathcal{Z}) = 0$ contradicting (5.7).

Proof of the proposition. Step 1. Existence of a negative eigenvalue. The fact that L is a self-adjoint operator on L^2 with domain H^2 is well-known. Note that $L = -\Delta - \frac{N+2}{N-2} W^{\frac{4}{N-2}}$ with $W^{\frac{4}{N-2}} \approx 1/|x|^4$ for large x . In particular, $W^{\frac{4}{N-2}}$ is bounded and tends to 0 at infinity, which shows that the essential spectrum of L is $[0, \infty)$ (see e.g. [RS78, Theorem XIII.14]). Furthermore, $|x| W^{\frac{4}{N-2}}$ tends to 0 at infinity, so that by Kato's Theorem [RS78, Theorem XII.58], L does not have any positive eigenvalue.

By the equation $-\Delta W = W^{\frac{N+2}{N-2}}$, we have $LW = -\frac{4}{N-2} W^{\frac{N+2}{N-2}}$ and thus

$$\int LWW = - \int \frac{4}{N-2} W^{2^*} < 0.$$

By approximating W by compactly supported functions

$$\inf_{\|u\|_2=1, u \in H^2} \int Luu < 0,$$

Thus L has at least one negative eigenvalue $-e_0^2$, and the corresponding eigenfunction \mathcal{Y} is exponentially decreasing by Agmon estimate. We chose $-e_0^2$ to be the first eigenvalue of L , which implies that \mathcal{Y} is radial and $-e_0^2$ is a simple eigenvalue

Step 2. Proof of (5.7). We first show

$$(5.8) \quad \forall f \in G_\perp, \quad Q(f) > 0.$$

If not $\exists f \in G_\perp, \quad Q(f) \leq 0$. Let $g \in \text{span}\{f, \mathcal{Y}, \widetilde{W}, W_1, \dots, W_N\}$. Taking into account that $L\widetilde{W} = LW_1 = \dots = LW_N = 0$, and that $\int L\mathcal{Y}f = -e_0^2 \int \mathcal{Y}f = 0$ we get

$$Q(g) = \int Lg g = \int L(\alpha\mathcal{Y} + \beta f)(\alpha\mathcal{Y} + \beta f) = \beta^2 Q(f) - \alpha^2 e_0^2 \int \mathcal{Y}^2 \leq 0.$$

Note that $\text{span}\{f, \mathcal{Y}, \widetilde{W}, W_1, \dots, W_N\}$ is a subspace of \dot{H}^1 of dimension $N+3$, whereas $H^\perp = \text{span}\{W, \widetilde{W}, W_1, W_2, \dots, W_N\}^\perp$ (the orthogonal is taken in \dot{H}^1) is of codimension $N+2$ in \dot{H}^1 . Thus there exists a nonzero $g \in \text{span}\{f, \mathcal{Y}, \widetilde{W}, W_1, \dots, W_N\} \cap H^\perp$. By Claim 3.5, $Q(g) > 0$, whereas we have just shown that $Q(g) \leq 0$ yielding a contradiction. The proof of (5.8) is complete.

We now turn to the proof of (5.7). We argue again by contradiction. If (5.7) does not hold, there exists a sequence (f_n) such that

$$(5.9) \quad f_n \in G_\perp, \quad \|\nabla f_n\|_2 = 1, \quad Q(f_n) \xrightarrow{n \rightarrow \infty} 0.$$

Extracting a subsequence from (f_n) , we may assume

$$f_n \rightharpoonup f \text{ in } \dot{H}^1.$$

The weak convergence of $f_n \in G_\perp$ to f implies that $f \in G_\perp$. By Cauchy-Schwarz inequality for the positive quadratic form Q on G_\perp , we get $\sqrt{Q(f_n)Q(f)} \geq |\int Lf f_n|$. Thus by (5.9),

$$Q(f) = \lim_{n \rightarrow +\infty} \int Lf f_n = 0.$$

As $f \in G_\perp$, (5.8) shows that $f = 0$ and thus

$$f_n \rightarrow 0 \text{ in } \dot{H}^1.$$

Now, by compactness $\int W^{\frac{4}{N-2}} |f_n|^2$ tends to 0. Using that by (5.9), $Q(f_n)$ tends to 0, we get that $\|\nabla f_n\|_2$ tends to 0, contradicting (5.9). The proof of (5.7) is complete.

Step 3. Uniqueness of the negative eigenvalue. Assume that L has a second eigenfunction \mathcal{Y}_1 , with eigenvalue $-e_1^2 \leq 0$. As $-e_0^2$ is the first eigenvalue of L , we have that $-e_0^2 < -e_1^2$ and $\int \mathcal{Y}\mathcal{Y}_1 = 0$. The same argument than above shows that

$$\forall f \in \text{span}\{\mathcal{Y}, \mathcal{Y}_1, \widetilde{W}, W_1, \dots, W_N\}, \quad Q(f) \leq 0,$$

which yields a subspace of \dot{H}^1 of dimension $N+3$ where Q is nonpositive, contradicting the fact that Q is positive on the subspace H^\perp , which is of codimension $N+2$ in \dot{H}^1 . \square

In the sequel, we will chose \mathcal{Y} such that

$$(5.10) \quad \int \mathcal{Y}^2 = 1.$$

5.3. Properties of the nonhomogeneous linearized equation. Let $t_0 \geq 0$. We are now interested by the following problem

$$(5.11) \quad \partial_t^2 h + Lh = \varepsilon, \quad t \geq t_0,$$

where $h \in C^0([t_0, +\infty), \dot{H}^1)$, $\partial_t h \in C^0([t_0, +\infty), L^2)$, $\varepsilon \in C^0([t_0, +\infty), L^{\frac{2N}{N+2}})$ and $D_x^{1/2} \varepsilon \in N(t_0, +\infty)$ (see (2.1) for the definition of $N(t_0, +\infty)$).

Proposition 5.7. *Let h and ε be as above. Assume that for some constant c_0, c_1 such that $0 < c_0 < c_1$,*

$$(5.12) \quad \|\partial_t h(t)\|_2 + \|\nabla h(t)\|_2 \leq Ce^{-c_0 t}$$

$$(5.13) \quad \|\varepsilon(t)\|_{\frac{2N}{N+2}} + \|D_x^{1/2} \varepsilon\|_{N(t, +\infty)} \leq Ce^{-c_1 t}.$$

Then, if c_1^- is any arbitrary number $< c_1$.

- *if $c_1 > e_0$, there exists $A \in \mathbb{R}$ such that*

$$(5.14) \quad \|\partial_t (h(t) - Ae^{-e_0 t} \mathcal{Y})\|_2 + \|(\nabla(h(t) - Ae^{-e_0 t} \mathcal{Y}))\|_2 \leq Ce^{-c_1^- t};$$

- *if $c_1 \leq e_0$,*

$$(5.15) \quad \|\partial_t h(t)\|_2 + \|\nabla h(t)\|_2 \leq Ce^{-c_1^- t}.$$

Proof. Write

$$h(t) = \beta(t) \mathcal{Y} + \tilde{\gamma}(t) \widetilde{W} + \sum_{j=1}^N \gamma_j(t) W_j + g(t), \quad g(t) \in G_\perp.$$

By the definition of G_\perp , the condition $g(t) \in G_\perp$ is equivalent to

$$(5.16) \quad \beta(t) := \int h(t) \mathcal{Y}, \quad \tilde{\gamma}(t) := \int \nabla(h(t) - \beta(t) \mathcal{Y}) \nabla \widetilde{W}, \quad \gamma_j(t) := \int \nabla(h(t) - \beta(t) \mathcal{Y}) \nabla W_j.$$

Step 1. Reduced case.

In this case, we assume, in addition to the hypothesis of Proposition 5.7

$$(5.17) \quad \forall t \geq t_0, \quad \beta(t) := \int h(t) \mathcal{Y} = 0.$$

And show that (5.14) (with $A = 0$) or (5.15) hold. It is sufficient to show

$$(5.18) \quad \|\partial_t h(t)\|_2 + \|\nabla h(t)\|_2 \leq Ce^{-\frac{(c_0 + c_1)}{2} t}.$$

An iteration argument will give the desired result.

We first prove

$$(5.19) \quad \frac{1}{2} \frac{d}{dt} (Q(h(t)) + \|\partial_t h(t)\|_2^2) = \int_{\mathbb{R}^N} \varepsilon(t) \partial_t h(t).$$

Indeed, recalling that $Q(h) = \int Lh h$, we get

$$\frac{1}{2} \frac{d}{dt} (Q(h(t)) + \|\partial_t h(t)\|_2^2) = \int_{\mathbb{R}^N} Lh(t) \partial_t h(t) + \int_{\mathbb{R}^N} \partial_t^2 h(t) \partial_t h(t),$$

which gives directly (5.19) from equation (5.11), .

We now turn to the proof of (5.18). Note that h is exponentially decreasing in the Strichartz norms:

$$(5.20) \quad \|h\|_{\ell(t,+\infty)} \leq Ce^{-c_0 t}$$

Indeed $\partial_t^2 h - \Delta h = \frac{N+2}{N-1} W^{\frac{4}{N+2}} h + \varepsilon$ and by Corollary 5.2, assumptions (5.12) and (5.13),

$$\left\| D_x^{1/2} \left(\frac{N+2}{N-1} W^{\frac{4}{N+2}} h + \varepsilon \right) \right\|_{N(t,+\infty)} \leq Ce^{-c_0 t}.$$

By Strichartz estimates (see Proposition 2.1), we get (5.20).

Now, by (5.19),

$$\left| \frac{d}{dt} (Q(h(t)) + \|\partial_t h(t)\|_2^2) \right| \leq C \|D_x^{1/2} \varepsilon(t)\|_{\frac{2(N+1)}{N+3}} \|D_x^{-1/2} \partial_t h(t)\|_{\frac{2(N+1)}{N-1}}.$$

Integrating between t and $+\infty$, we get, combining assumption (5.13) on ε , estimate (5.20), and Hölder inequality in time,

$$Q(h(t)) + \|\partial_t h(t)\|_2^2 \leq C \|D_x^{1/2} \varepsilon\|_{N(t,+\infty)} \|h\|_{\ell(t,+\infty)} \leq Ce^{-(c_0+c_1)t}.$$

By Claim 3.5, and (5.17), $\|\nabla g(t)\|_2^2 \leq CQ(g(t)) = CQ(h(t)) \leq Ce^{-(c_0+c_1)t}$. Hence

$$(5.21) \quad \|\partial_t h(t)\|_2 + \|\nabla g(t)\|_2 \leq Ce^{-\frac{(c_0+c_1)}{2}t}.$$

Furthermore, note that by the definition of $\tilde{\gamma}$ in (5.16) and (5.21).

$$\tilde{\gamma}(t) = \int_{\mathbb{R}^N} h(t) \Delta \widetilde{W} \xrightarrow[t \rightarrow +\infty]{} 0, \quad |\tilde{\gamma}'(t)| = \left| \int_{\mathbb{R}^N} \partial_t h(t) \Delta \widetilde{W} \right| \leq Ce^{-\frac{c_0+c_1}{2}t}.$$

Hence

$$(5.22) \quad |\tilde{\gamma}(t)| \leq Ce^{-\frac{(c_0+c_1)}{2}t}.$$

By an analogous argument

$$(5.23) \quad \sum_{j=1}^N |\gamma_j(t)| \leq Ce^{-\frac{(c_0+c_1)}{2}t}.$$

This gives (5.18) and concludes Step 1.

Step 2. General case. We no longer assume $\beta(t) = 0$. We have:

$$(5.24) \quad \beta''(t) = e_0^2 \beta(t) + \eta(t), \quad \text{where } \eta(t) := \int_{\mathbb{R}^N} \varepsilon(t) \mathcal{Y}.$$

Indeed,

$$\beta''(t) = \int_{\mathbb{R}^N} \partial_t^2 h(t) \mathcal{Y} = - \int_{\mathbb{R}^N} Lh(t) \mathcal{Y} + \int_{\mathbb{R}^N} \varepsilon(t) \mathcal{Y} = e_0^2 \int_{\mathbb{R}^N} h(t) \mathcal{Y} + \eta(t) \mathcal{Y}.$$

We will show that $\tilde{h}(t) := h(t) - \beta(t) \mathcal{Y}$ and $\tilde{\varepsilon}(t) := \varepsilon(t) - \eta(t) \mathcal{Y}$ satisfy the hypothesis of Step 1.

By assumption (5.13), $|\eta(t)| \leq Ce^{-c_1 t}$. We distinguish two cases.

First case: $e_0 < c_1$. Then $e^{e_0 t} |\eta(t)| \leq Ce^{-(c_1-e_0)t}$ with $c_1 - e_0 > 0$. Solving (5.24), we see that there exist real parameters β_+ , β_- such that

$$\beta(t) = \beta_- e^{-e_0 t} + \beta_+ e^{e_0 t} - \frac{1}{2e_0} \int_t^{+\infty} e^{e_0(t-s)} \eta(s) ds + \frac{1}{2e_0} \int_t^{+\infty} e^{-e_0(t-s)} \eta(s) ds.$$

Note that $\left| \int_t^{+\infty} e^{e_0(t-s)} \eta(s) ds \right| + \left| \int_t^{+\infty} e^{-e_0(t-s)} \eta(s) ds \right| \leq C e^{-c_1 t}$. Furthermore $\beta(t)$ tends to 0 by (5.12) which shows that $\beta_+ = 0$. Hence

$$(5.25) \quad \beta(t) = \beta_- e^{-e_0 t} + O(e^{-c_1 t}).$$

Second case: $c_1 \leq e_0$. Solving again (5.24), we get real parameters β_+ , β_- such that

$$\beta(t) = \beta_- e^{-e_0 t} + \beta_+ e^{e_0 t} - \frac{1}{2e_0} \int_t^{+\infty} e^{e_0(t-s)} \eta(s) ds - \frac{1}{2e_0} \int_0^t e^{-e_0(t-s)} \eta(s) ds.$$

Note that

$$\left| \frac{1}{2e_0} \int_t^{+\infty} e^{e_0(t-s)} \eta(s) ds + \frac{1}{2e_0} \int_0^t e^{-e_0(t-s)} \eta(s) ds \right| \leq C \begin{cases} e^{-c_1 t} & \text{if } c_1 < e_0 \\ t e^{-c_1 t} & \text{if } c_1 = e_0 \end{cases},$$

so that we must have again $\beta_+ = 0$. As a conclusion

$$(5.26) \quad c_1 < e_0 \implies \beta(t) = O(e^{-c_1 t}), \quad c_1 = e_0 \implies \beta(t) = O(t e^{-c_1 t}).$$

In view of (5.24), it is easy to check that in both cases, \tilde{h} and $\tilde{\varepsilon}$ satisfy the assumptions of Step 1, which implies, together with (5.25) or (5.26), the conclusions (5.14) or (5.15) of Proposition 5.7. \square

6. PROOF OF MAIN RESULTS

In this section we conclude the proofs of Theorem 1 and 2. We start, in Subsection 6.1, by constructing approximate solutions U_k^a of (1.1) which converge to W as $t \rightarrow +\infty$. Subsection 6.2 is devoted to a fixed point argument near U_k^a for large k . The proof of Theorems 1 and 2 is the object of Subsection 6.3, except for the blow-up of W^+ for negative times, which is shown in Subsection 6.4.

6.1. A family of approximate solutions converging to W .

Lemma 6.1. *Let $a \in \mathbb{R}$. There exist functions $(\Phi_j^a)_{j \geq 1}$ in $\mathcal{S}(\mathbb{R}^N)$, such that $\Phi_1^a = a\mathcal{Y}$ and if*

$$(6.1) \quad U_k^a(t, x) := W(x) + \sum_{j=1}^k e^{-j e_0 t} \Phi_j^a(x),$$

then as $t \rightarrow +\infty$,

$$(6.2) \quad \varepsilon_k := \partial_t^2 U_k^a - \Delta U_k^a - |U_k^a|^{\frac{4}{N-2}} U_k^a = O(e^{-(k+1)e_0 t}) \text{ in } \mathcal{S}(\mathbb{R}^N).$$

Proof. The proof is identical to the proof of [DM07, Lemma 6.1]. We sketch it for the sake of completeness. Note that

$$(6.3) \quad \partial_t^2(W + h) - \Delta(W + h) + |W + h|^{\frac{4}{N-2}}(W + h) = \partial_t^2 h + Lh - R(h)$$

where L and R are defined in (5.1). We have

$$R(h) = W^{\frac{N+2}{N-2}} J(W^{-1}h), \quad J(t) := |1+t|^{\frac{4}{N-2}}(1+t) - 1 - \frac{N+2}{N-2}t.$$

The function J is real analytic for $|t| < 1$ and $J(0) = J'(0) = 0$. Thus $J(t)$ is a power series of the form $\sum_{j \geq 2} c_j t^j$ which has radius of convergence 1. In particular, if $h \in \mathcal{S}(\mathbb{R}^N)$ and satisfies $|h(x)W^{-1}(x)| \leq 1/2$, for all $x \in \mathbb{R}^N$, then

$$(6.4) \quad R(h) = \sum_{j=2}^{+\infty} c_j W^{\frac{N+2}{N-2}} (hW^{-1})^j,$$

where the series converges in $\mathcal{S}(\mathbb{R}^N)$.

Let us fix $a \in \mathbb{R}$. We will omit most superscripts a to simplify notations. Clearly, by (6.3), if $U_1 = W + a\mathcal{Y}e^{-e_0 t}$,

$$\partial_t^2 U_1 - \Delta U_1 - |U_1|^{\frac{4}{N-2}} U_1 = e_0^2 a e^{-e_0 t} \mathcal{Y} + a e^{-e_0 t} L(\mathcal{Y}) - R(a\mathcal{Y}e^{-e_0 t}) = -R(a\mathcal{Y}e^{-e_0 t}),$$

which yields (6.2) for $k = 1$.

Now assume that for some $k \geq 1$, there exist Φ_1, \dots, Φ_k in $\mathcal{S}(\mathbb{R}^N)$ such that (6.2) holds with

$$(6.5) \quad U_k = W + h_k, \text{ where } h_k := \sum_{j=1}^k e^{-j e_0 t} \Phi_j.$$

Let $\Phi_{k+1} \in \mathcal{S}(\mathbb{R}^N)$. Then

$$(6.6) \quad \begin{aligned} \partial_t^2 \left(h_k + e^{-(k+1)e_0 t} \Phi_{k+1} \right) + L \left(h_k + e^{-(k+1)e_0 t} \Phi_{k+1} \right) - R \left(h_k + e^{-(k+1)e_0 t} \Phi_{k+1} \right) \\ = ((k+1)^2 e_0^2 + L) e^{-(k+1)e_0 t} \Phi_{k+1} + \varepsilon_k + R(h_k) - R(h_k + e^{-(k+1)e_0 t} \Phi_{k+1}). \end{aligned}$$

By (6.4) we see that ε_k must be, for large $t > 0$, an infinite sum of the form $\sum_{j \geq 0} e^{-j e_0 t} \Psi_{j,k}(x)$, with convergence in $\mathcal{S}(\mathbb{R}^N)$. Furthermore, the induction hypothesis (6.2) shows that $\Psi_{j,k} = 0$ for $j \leq k$. Thus

$$\varepsilon_k(t, x) = \sum_{j \geq k+1} e^{-j e_0 t} \Psi_{j,k}(x).$$

Furthermore, $R(h_k)$ and $R(h_k + e^{-(k+1)e_0 t} \Phi_{k+1})$ have similar asymptotic developments, and if t is large enough, using that $h_k = O(e^{-e_0 t})$, we get $|R(h_k) - R(h_k + e^{-(k+1)e_0 t} \Phi_{k+1})| \leq C e^{-e_0(k+2)t}$. This shows

$$(6.7) \quad \varepsilon_k + R(h_k) - R(h_k + e^{-(k+1)e_0 t} \Phi_{k+1}) = e^{-(k+1)e_0 t} \Psi_{k+1,k} + O(e^{-(k+2)e_0 t}) \text{ in } \mathcal{S}(\mathbb{R}^N).$$

By Proposition 5.5, $-(k+1)^2 e_0^2$ is not in the spectrum of L . It is classical that the resolvent $((k+1)^2 e_0^2 + L)^{-1}$ maps \mathcal{S} into \mathcal{S} (see e.g. [DM07, §7.2.2] for the proof of a similar fact). In view of (6.6) and (6.7), it suffices to take

$$\Phi_{k+1} := ((k+1)^2 e_0^2 + L)^{-1} \Psi_{k+1,k}$$

to get (6.2) at rank $k+1$. □

6.2. Contraction argument near an approximate solution of large order.

Proposition 6.2. *Let $a \in \mathbb{R}$. There exists $k_0 > 0$ such that for any $k \geq k_0$, there exists $t_k \geq 0$ and a solution U^a of (1.1) such that for $t \geq t_k$,*

$$(6.8) \quad \|U^a - U_k^a\|_{\ell(t, +\infty)} \leq e^{-(k+\frac{1}{2})e_0 t}.$$

Furthermore, U^a is the unique solution of (1.1) satisfying (6.8) for large t . It is independent of k and satisfies, for large t ,

$$(6.9) \quad \|\nabla(U^a(t) - U_k^a)\|_2 + \|\partial_t(U^a(t) - U_k^a)\|_2 \leq e^{-(k+\frac{1}{2})e_0 t}.$$

Proof. We follow the lines of the proof of [DM07, Proposition 6.3].

Step 1. Transformation into a fixed-point problem. As in the proof of Lemma 6.1, we will fix $a \in \mathbb{R}$ and omit most of the superscripts a . Recall the definition of ε_k and h_k from (6.1) and (6.5). The proof relies on a fixed point argument to construct

$$w := U^a - W - h_k.$$

The function U^a is solution of (1.1) if and only if $U^a - W$ is solution of (5.1). Subtracting equation (5.1) on $U^a - W$ and the equation $\partial_t^2 h_k + Lh_k = R(h_k) + \varepsilon_k$, we get that U^a satisfies (1.1) if and only if w satisfies $\partial_t^2 w + Lw = R(h_k + w) - R(h_k) - \varepsilon_k$. This may be written as

$$\partial_t^2 w - \Delta w = \frac{N+2}{N-2} W^{\frac{4}{N-2}} w + R(h_k + w) - R(h_k) - \varepsilon_k.$$

Thus the existence of a solution U^a of (1.1) satisfying (6.8) for $t \geq t_k$ may be written as the following fixed-point problem on w

$$(6.10) \quad \forall t \geq t_k, \quad w(t) = \mathcal{M}_k(w)(t) \text{ and } \|w\|_{\ell(t, +\infty)} \leq e^{-(k+\frac{1}{2})e_0 t}, \text{ where } \mathcal{M}_k(w)(t) := \\ - \int_t^{+\infty} \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} \left[\frac{N+2}{N-2} W^{\frac{4}{N-2}} w(s) + R(h_k(s) + w(s)) - R(h_k(s)) - \varepsilon_k(s) \right] ds.$$

Let us fix k and t_k . Consider

$$E_\ell^k := \left\{ w \in \ell(t_k, +\infty); \|w\|_{E_\ell^k} := \sup_{t \geq t_k} e^{(k+\frac{1}{2})e_0 t} \|w\|_{\ell(t, +\infty)} < \infty \right\} \\ B_\ell^k := \{ w \in E_\ell^k, \|w\|_{E_\ell^k} \leq 1 \}.$$

The space E_ℓ^k is clearly a Banach space. In view of (6.10), it is sufficient to show that if t_k and k are large enough, the mapping \mathcal{M}_k is a contraction on B_ℓ^k .

Step 2. Contraction property. Note that by Strichartz inequality on the free equation (Lemma 2.5), there is a constant $C^* > 0$ such that if $w, \tilde{w} \in E_\ell^k$, $k \geq 1$,

$$(6.11) \quad \|\mathcal{M}_k(w)\|_{\ell(t,+\infty)} \leq C^* \left[\left\| D_x^{1/2} \left(W^{\frac{4}{N-2}} w \right) \right\|_{N(t,+\infty)} + \left\| D_x^{1/2} (R(h_k + w) - R(h_k)) \right\|_{N(t,+\infty)} + \|D_x^{1/2} \varepsilon_k\|_{N(t,+\infty)} \right]$$

$$(6.12) \quad \|\mathcal{M}_k(w) - \mathcal{M}_k(\tilde{w})\|_{\ell(t,+\infty)} \leq C^* \left[\left\| D_x^{1/2} \left(W^{\frac{4}{N-2}} (w - \tilde{w}) \right) \right\|_{N(t,+\infty)} + \left\| D_x^{1/2} (R(h_k + w) - R(h_k + \tilde{w})) \right\|_{N(t,+\infty)} \right].$$

Claim 6.3. *There exists a constant $C_k > 0$, depending only on k such that for all $w, \tilde{w} \in B_\ell^k$ and $t \geq t_k$*

$$(6.13) \quad \|D_x^{1/2} \varepsilon_k\|_{N(t,+\infty)} \leq C_k e^{-(k+1)e_0 t},$$

$$(6.14) \quad \|D_x^{1/2} (R(h_k + w) - R(h_k + \tilde{w}))\|_{N(t,+\infty)} \leq C_k e^{-(k+\frac{3}{2})e_0 t} \|w - \tilde{w}\|_{E_\ell^k}.$$

Furthermore, there exists $k_0 > 0$ such that for all $k \geq k_0$ and all $w \in E_\ell^k$

$$(6.15) \quad \left\| D_x^{1/2} \left(W^{\frac{4}{N-2}} w \right) \right\|_{N(t,+\infty)} \leq \frac{1}{4C^*} e^{-(k+\frac{1}{2})e_0 t} \|w\|_{E_\ell^k}.$$

Proof of Claim 6.3. The proof is very close to [DM07, Claim 6.4]. Estimate (6.13) follows immediately from (6.2), and (6.14) follows immediately from Corollary 5.2.

Let us show (6.15). Let $\tau_0 \in (0, 1)$. By Lemma 5.1, there exists a constant $C_2 > 0$ such that

$$\|D_x^{1/2} (W^{\frac{4}{N-2}} w)\|_{N(t, t+\tau_0)} \leq C_2 \tau_0^{\frac{2}{N+1}} \|w\|_{\ell(t, t+\tau_0)} \leq C_2 \tau_0^{\frac{2}{N+1}} e^{-(k+\frac{1}{2})e_0 t} \|w\|_{E_\ell^k}.$$

By Claim 5.3,

$$\|D_x^{1/2} (W^{\frac{4}{N-2}} w)\|_{N(t,+\infty)} \leq \frac{C_2 e^{-(k+\frac{1}{2})e_0 t}}{1 - e^{-(k+\frac{1}{2})e_0 \tau_0}} \tau_0^{\frac{2}{N+1}} \|w\|_{E_\ell^k}.$$

Choosing τ_0 and k_0 such that $C_2 \tau_0^{\frac{2}{N+1}} = \frac{1}{8C^*}$, and $e^{-(k_0+\frac{1}{2})e_0 \tau_0} \leq \frac{1}{2}$, we get (6.15) for $k \geq k_0$. \square

Chose $k \geq k_0$. By (6.11) and Claim 6.3, we get, if $g \in B_\ell^k$

$$\begin{aligned} \|\mathcal{M}_k(w)\|_{\ell(t,+\infty)} &\leq \left(\frac{1}{4} e^{-(k+\frac{1}{2})e_0 t} \|w\|_{E_\ell^k} + C^* C_k e^{-(k+\frac{3}{2})e_0 t} \|w\|_{E_\ell^k} + C^* C_k e^{-(k+1)e_0 t} \right) \\ &\leq e^{-(k+\frac{1}{2})e_0 t} \left(\frac{1}{4} + C^* C_k e^{-e_0 t_k} + C^* C_k e^{-\frac{1}{2}e_0 t_k} \right). \end{aligned}$$

Choosing t_k so large that $C^* C_k e^{-e_0 t_k} + C^* C_k e^{-\frac{1}{2}e_0 t_k} \leq \frac{1}{2}$, we get that for large k , $\mathcal{M}_k(w)$ is in B_ℓ^k .

Now, let $w, \tilde{w} \in B_\ell^k$. Similarly, by (6.12) and Claim 6.3,

$$\|\mathcal{M}_k(w) - \mathcal{M}_k(\tilde{w})\|_{E_\ell^k} \leq \|w - \tilde{w}\|_{E_\ell^k} \left(\frac{1}{4} + C^* C_k e^{-e_0 t_k} \right),$$

which shows, choosing a larger t_k if necessary, that \mathcal{M}_k is a contraction of B_ℓ^k .

Thus, for each $k \geq k_0$, (1.1) has a unique solution U^a satisfying (6.8) for $t \geq t_k$. The preceding proof clearly remains valid taking a larger t_k , so that the uniqueness still holds in the class of solutions of (1.1) satisfying (6.8) for $t \geq t'_k$, where t'_k is any real number larger than t_k . Using the uniqueness in the Cauchy problem (1.1), it is now straightforward to show that U^a does not depend on $k \geq k_0$.

It remains to show (6.9). Let $k > 0$ be a large integer and $w \in B_\ell^k$. By Strichartz inequality (2.6), and the definition of \mathcal{M}_k , we have, for $t \geq t_k$,

$$\|\nabla \mathcal{M}_k(w)(t)\|_2 + \|\partial_t \mathcal{M}_k(w)(t)\|_2 \leq \left\| D_x^{1/2} \left(\frac{N+2}{N-2} W^{\frac{4}{N-2}} w + R(h_k + w) - R(h_k) - \varepsilon_k \right) \right\|_{N(t, +\infty)}.$$

As a consequence of Claim 6.3 and the fact that $\|w\|_{E_\ell}^k \leq 1$, we get

$$\|\mathcal{M}_k(w)(t)\|_{\dot{H}^1} \leq C \left(e^{-(k+\frac{1}{2})e_0 t} \|w\|_{E_\ell^k} + e^{-(k+1)e_0 t} \right) \leq C e^{-(k+\frac{1}{2})e_0 t}.$$

Applying the preceding inequality to the solution $w = U^a - U_k^a$ of the fixed point $w = \mathcal{M}_k(w)$, we get directly (6.9). The proof of Proposition 6.2 is complete. \square

6.3. Conclusion of the proofs. At this levels, the proof are similar to the one of [DM07], except for the blowing-up of W^+ which is proven in the next subsection.

Proof of Theorem 1. The function \mathcal{Y} is an eigenfunction for the first eigenvalue of L , thus it must have constant sign. Replacing \mathcal{Y} by $-\mathcal{Y}$ if necessary, we may assume that $\mathcal{Y}(x) > 0$ for all x and thus

$$(6.16) \quad \int \nabla \mathcal{Y} \cdot \nabla W > 0.$$

Let

$$W^\pm := U^{\pm 1},$$

which yields two solutions of (1.1) for large $t \geq t_0$. Then all the conditions of Theorem 1 are satisfied. Indeed, W^\pm is globally defined for $t \geq t_0$ and by (6.9), $(W^\pm, \partial_t W^\pm)$ tends to $(W, 0)$ in $\dot{H}^1 \times L^2$, which yields (1.7). The energy condition (1.6) then follows from the conservation of the energy. Furthermore, again by (6.9),

$$\|\nabla U^a\|_2^2 = \|\nabla W\|_2^2 + 2ae^{-e_0 t} \int (\nabla W \cdot \nabla \mathcal{Y} + O(e^{-\frac{3}{2}e_0 t})),$$

which shows, together with (6.16), that for large $t > 0$,

$$\|\nabla W^+(t)\|_2 > 0, \quad \|\nabla W^-(t)\|_2 < 0.$$

From Remark 1.3, this inequalities remain valid for every t in the intervals of existence of W^+ and W^- .

Finally $T_-(W^-) = -\infty$ by (a) in Proposition 2.8 and $\|u\|_{S(-\infty, 0)} < \infty$ by (3.2) in Proposition 3.1.

Except for the proof of the finite time blow-up of W^+ for negative time, which we postpone to Subsection 6.4, the proof of Theorem 1 is complete. \square

Proof of Theorem 2. Let us first prove:

Lemma 6.4. *If u is a solution of (1.1) satisfying*

$$(6.17) \quad \|\nabla(u(t) - W)\|_2 + \|\partial_t u(t)\|_2 \leq Ce^{-\gamma_0 t}, \quad E(u_0, u_1) = E(W)$$

then

$$\exists! a \in \mathbb{R}, \quad u = U^a.$$

Corollary 6.5. *For any $a \neq 0$, there exists $T_a \in \mathbb{R}$ such that*

$$(6.18) \quad \begin{cases} U^a = W^+(t + T_a) & \text{if } a > 0 \\ U^a = W^-(t + T_a) & \text{if } a < 0. \end{cases}$$

Proof of Lemma 6.4. Let $u = W + h$ be a solution of (1.1) for $t \geq t_0$ satisfying (6.17). Recall that h satisfies equation (5.1).

Step 1. We show that there exists $a \in \mathbb{R}$ such that

$$(6.19) \quad \forall \eta > 0, \quad \|\nabla(h(t) - ae^{-e_0 t} \mathcal{Y})\|_2 + \|\partial_t h(t) + ae_0 e^{-e_0 t} \mathcal{Y}\|_2 + \|h(s) - ae^{-e_0 s} \mathcal{Y}\|_{\ell(t, +\infty)} \leq C_\eta e^{-(2-\eta)e_0 t}.$$

Indeed we will show

$$(6.20) \quad \|\nabla h(t)\|_2 + \|\partial_t h\|_2 \leq Ce^{-e_0 t}, \quad \|R(h(t))\|_{L^{\frac{2N}{N+2}}} + \|D_x^{1/2}(R(h))\|_{N(t, +\infty)} \leq Ce^{-2e_0 t}.$$

Assuming (6.20), we are in the setting of Proposition 5.7, with $\varepsilon = R(h)$, $c_0 = e_0$ and $c_1 = 2e_0$. The conclusion (5.15) of the lemma would then yield (6.19). It remains to prove (6.20).

By Corollaries 5.2 and 5.4 the bound on $R(h)$ in (6.20) follows from the bound on $\|\nabla h(t)\|_2 + \|\partial_t h(t)\|_2$, so that we only need to show this first bound. By Corollary 5.4, assumption (6.17) implies $\|\nabla h(t)\|_2 + \|\partial_t h(t)\|_2 + \|h\|_{\ell(t, +\infty)} \leq Ce^{-\gamma_0 t}$. By Corollary 5.2

$$\|R(h(t))\|_{L^{\frac{2N}{N+2}}} + \|\nabla(R(h))\|_{N(t, +\infty)} \leq Ce^{-2\gamma_0 t}.$$

Thus we can apply Proposition 5.7, showing that

$$\|h(t)\|_{\dot{H}^1} \leq C(e^{-e_0 t} + e^{-\frac{3}{2}\gamma_0 t}).$$

If $\frac{3}{2}\gamma_0 \geq e_0$ the proof of (6.20) is complete. If not, assumption (6.17) on v holds with $\frac{3}{2}\gamma_0$ instead of γ_0 , and an iteration argument yields (6.20).

Step 2. Let us show

$$(6.21) \quad \forall m > 0, \exists t_0 > 0, \forall t \geq t_0, \quad \|\partial_t(u(t) - U^a(t))\|_2 + \|\nabla(u - U^a)\|_{\ell(t, +\infty)} \leq e^{-mt}.$$

By Step 1, (6.21) holds for $m = \frac{3}{2}e_0$. Let us assume (6.21) holds for some $m = m_1 > e_0$. We will show that it holds for $m = m_1 + \frac{e_0}{2}$, which will yield (6.21) by iteration and conclude the proof.

Write $h(t) := u(t) - W$, $h^a(t) := U^a(t) - W$ (so that in particular $u - U^a = h - h^a$). Then

$$\partial_t^2(h - h^a) + L(h - h^a) = -R(h) + R(h^a).$$

By induction hypothesis $\|\partial_t(h(t) - h^a(t))\|_2 + \|\nabla(h(t) - h^a(t))\|_2 + \|\nabla(h - h^a)\|_{\ell(t, +\infty)} \leq e^{-m_1 t}$. According to Corollary 5.2

$$\|D_x^{1/2}(R(h) - R(h^a))\|_{N(t, +\infty)} + \|R(h(t)) - R(h^a(t))\|_{L^{\frac{2N}{N+2}}} \leq Ce^{-(m_1 + e_0)t}.$$

Then by Proposition 5.7

$$\|\partial_t(h(t) - h^a(t))\|_2 + \|\nabla(h(t) - h^a(t))\|_2 + \|\nabla(h - h^a)\|_{\ell(t, +\infty)} \leq Ce^{-(m_1 + \frac{3}{4}e_0)t},$$

which yields (6.21) with $m = m_1 + \frac{e_0}{2}$. By iteration, (6.21) holds for any $m > 0$. Taking $m = (k_0 + 1)e_0$ (where k_0 is given by Proposition 6.2), we get that for large $t > 0$

$$\|\nabla(u - U_{k_0}^a)\|_{Z(t, +\infty)} \leq e^{-(k_0 + \frac{1}{2})e_0 t}.$$

By uniqueness in Proposition 6.2, we get as announced that $u = W^a$ which concludes the proof of the lemma. \square

Proof of Corollary 6.5. Let $a \neq 0$ and chose T_a such that $|a|e^{-e_0 T_a} = 1$. By (6.9),

$$(6.22) \quad \|\partial_t(U^a(t + T_a) - W \mp e^{-e_0 t} \mathcal{Y})\|_2 + \|\nabla(U^a(t + T_a) - W \mp e^{-e_0 t} \mathcal{Y})\|_2 \leq Ce^{-\frac{3}{2}e_0 t}.$$

Furthermore, $U^a(\cdot + T_a)$ satisfies the assumptions of Lemma 6.4, which shows that there exists $a' \in \mathbb{R}$ such that $U^a(\cdot + T_a) = U^{a'}$. By (6.22), $a' = 1$ if $a > 0$ and $a' = -1$ if $a < 0$, hence (6.18). \square

Let us turn to the proof of Theorem 2. Point (b) is an immediate consequence of the variational characterization of W ([Aub76], [Tal76]).

Let us show (a). Let u be a solution of (1.1) such that $E(u_0, u_1) = E(W, 0)$ and $\|\nabla u_0\|_2 < \|\nabla W\|_2$. Assume that $\|u\|_{S(\mathbb{R})} = \infty$. Replacing if necessary $u(t)$ by $u(-t)$, we may assume that $\|u\|_{S(0, +\infty)} = \infty$. Then (replacing u by $-u$ if necessary), Proposition 3.1 shows that there exist $\mu_0 > 0$, $x_0 \in \mathbb{R}^N$, and $c, C > 0$ such that $\|\partial_t u(t)\|_2 + \|\nabla(u(t) - W_{\mu_0, x_0})\|_2 \leq Ce^{-ct}$. This shows that $u_{\mu_0^{-1}, -\mu_0^{-1}x_0}$ fullfills the assumptions of Lemma 6.4 with $\|\nabla u_0\|_2 < \|\nabla W\|_2$. Thus there exists $a < 0$ such that $u_{\mu_0^{-1}, -\mu_0^{-1}x_0} = U^a$. By Corollary 6.5,

$$u(t) = W_{\mu_0, x_0}^-(t + T_a),$$

which shows (a).

The proof of (c) is similar. Indeed if u is a solution of (1.1) defined on $[0, +\infty)$ and such that $E(u_0, u_1) = E(W, 0)$, $\|\nabla u_0\|_2 > \|\nabla W\|_2$ and $u_0 \in L^2$, then by Proposition 4.1, $\|u(t) - W_{\mu_0, x_0}\|_{\dot{H}^1} \leq Ce^{-ct}$, which shows using Lemma 6.4 and the same argument as before that for some $T \in \mathbb{R}$,

$$u(t) = W_{\mu_0, x_0}^+(t + T).$$

The proof of Theorem 2 is complete. \square

6.4. Blow-up of W^+ . In this section we prove that the function W^+ blows-up in finite negative time.

We will argue by contradiction, assuming that W^+ is globally defined. As before, we will write $d(t) := \int |\nabla W^+|^2 - \int |\nabla W|^2 + \int |\partial_t W^+|^2$. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$, radial such that $0 \leq \varphi(x) \leq 1$, $\varphi(x) = 1$ if $|x| \leq 1$ and $\varphi(x) = 0$ if $|x| \geq 2$. Let $\varphi_R(x) = \varphi(x/R)$. Consider

$$y_R(t) := \int (W^+)^2 \varphi_R.$$

We start with some estimates on y_R' and y_R'' .

Step 1. Estimates for large positive t . Let us show that there exists $R_0, t_0, c_0 > 0$ such that for all $R \geq R_0$,

$$(6.23) \quad \forall t \geq t_0, \quad y_R''(t) \geq 4 \frac{N-1}{N-2} \int (\partial_t W^+)^2 + \frac{2}{N-2} \left(\int |\nabla W^+|^2 - |\nabla W|^2 \right) \geq \frac{2}{N-2} d(t).$$

$$(6.24) \quad y_R'(t_0) \leq -2c_0, \quad y_R(t_0) \leq \begin{cases} \frac{C}{R} & \text{if } N = 3 \\ C \log R & \text{if } N = 4 \\ C & \text{if } N = 5 \end{cases}$$

By explicit computations and $E(W^+, \partial_t W^+) = E(W, 0)$, we have

$$(6.25) \quad y_R'(t) = 2 \int W^+ \partial_t W^+ \varphi_R,$$

$$(6.26) \quad y_R''(t) = 4 \frac{N-1}{N-2} \int_{\mathbb{R}^N} (\partial_t W^+)^2 + \frac{4}{N-2} \left(\int_{\mathbb{R}^N} |\nabla W^+|^2 - \int_{\mathbb{R}^N} |\nabla W|^2 \right) \\ + \int \Delta \varphi_R (W^+)^2 + 2 \int (1 - \varphi_R) \left(|\nabla W^+|^2 - |W^+|^{2*} - |\partial_t W^+|^2 \right).$$

Replacing W by W^+ in the preceding expressions, we see that the corresponding y_R must be constant, so that in particular,

$$\int \Delta \varphi_R (W)^2 + 2 \int (1 - \varphi_R) \left(|\nabla W|^2 - |W|^{2*} \right) = 0.$$

By (6.8) in Proposition 6.2, $W^+ = W + e^{-e_0 t} \mathcal{Y} + r_1$ with $\|\nabla r_1\|_2 + \|\partial_t r_1\|_2 \leq C e^{-2e_0 t}$. Developping W^+ , we get, recalling that \mathcal{Y} is in \mathcal{S} and that $\varphi_R(x) = 1$ for $|x| \leq R$,

$$\left| \int \Delta \varphi_R (W^+)^2 + 2 \int (1 - \varphi_R) \left(|\nabla W^+|^2 - |W^+|^{2*} - |\partial_t W^+|^2 \right) \right| \leq C \left(\frac{e^{-e_0 t}}{R} + e^{-2e_0 t} \right), \\ \int |\nabla W^+(t)|^2 - \int |\nabla W|^2 = 2e^{-e_0 t} \int W \mathcal{Y} + O(e^{-2e_0 t}).$$

Thus by (6.26)

$$y_R''(t) \geq 4 \frac{N-1}{N-2} \int (\partial_t W^+)^2 + \frac{2}{N-2} \left[\int |\nabla W^+(t)|^2 - \int |\nabla W|^2 \right] \\ + \frac{4}{N-2} e^{-e_0 t} \int W \mathcal{Y} - C \left(\frac{e^{-e_0 t}}{R} + e^{-2e_0 t} \right).$$

As $\int W \mathcal{Y} > 0$, we get (6.23) for $R \geq R_0$.

Now, fixing $R \geq R_0$, we get, using that $y_R'(t)$ tends to 0 at infinity,

$$y_R'(t_0) = - \int_{t_0}^{+\infty} y_R''(t) dt \leq -4 \frac{N-1}{N-2} \int_{t_0}^{+\infty} |\partial_t W^+(t)|^2 dt,$$

which yields the first assertion in (6.24).

It remains to show the second assertion in (6.24). Note that $W \approx \frac{C}{|x|^{N-2}}$ at infinity, so that

$$(6.27) \quad \lim_{t \rightarrow +\infty} y_R(t) = \int W^2 \varphi_R \approx \begin{cases} R & \text{if } N = 3, \\ \log R & \text{if } N = 4, \\ \frac{1}{R} & \text{if } N = 5. \end{cases}$$

Furthermore, $|y'_R(t)| \leq C \|\partial_t W^+\|_2 \sqrt{y_R(t)} \leq C e^{-e_0 t} \sqrt{y_R(t)}$, and thus

$$\sqrt{y_R(t_0)} - \lim_{t \rightarrow \infty} \sqrt{y_R(t)} \leq C \int_{t_0}^{\infty} e^{-e_0 t} dt \leq C,$$

which yields together with (6.27), the second assertion in (6.24). Step 1 is complete

Step 2. Estimates for $t_0 - \varepsilon_0 R \leq t \leq t_0$. As a consequence of the preceding estimates, we show that there exists $C_0 > 0$ such that for $R \geq R_0$ and $t_0 - \varepsilon_0 R \leq t \leq t_0$,

$$(6.28) \quad y''_R(t) \geq 4 \frac{N-1}{N-2} \int (\partial_t W^+)^2 + \frac{4}{N-2} \left(\int |\nabla W^+|^2 - \int |\nabla W|^2 \right) - \frac{C_0}{R^{N-2}},$$

$$(6.29) \quad y'_R(t) \leq -c_0,$$

where $\varepsilon_0 := \frac{c_0}{2C_0}$.

Estimate (6.29) follows from (6.28) by integration in time and the fact that $y'_R(t_0) \leq -2c_0$ for R large. Let us show (6.28).

By (6.26), it is sufficient to show

$$(6.30) \quad \int_{|x| \geq R} r(W^+)(t) dx \leq \frac{C}{R^{N-2}}.$$

where $r(W^+)$ is defined in (2.25). Writing $W^+ = W + O(e^{-e_0 t})$, and using that $W \approx \frac{1}{|x|^{N-2}}$, $|\nabla W| \approx \frac{1}{|x|^{N-1}}$, as $|x| \rightarrow +\infty$, we get for large R

$$\int_{|x| \geq \frac{R}{6}} r(W^+)(t_0 + R/4) \leq \frac{C}{R^{N-2}} + C e^{-\frac{R}{4} e_0} \leq \frac{C}{R^{N-2}}.$$

Hence by finite speed of propagation (Proposition 2.3 (d)), and taking R large,

$$(6.31) \quad \forall t \leq t_0, \quad \int_{|x| \geq \frac{R}{2} + t_0 - t} r(W^+)(t) \leq \frac{C}{R^{N-2}},$$

which yields (6.30), and thus (6.28).

Step 3. Differential inequalities. Let us show that there exists a constant $C > 0$ such that

$$(6.32) \quad \forall t \in \left[t_0 - \frac{\varepsilon_0}{2} R, t_0 \right], \quad 0 \leq -y'_R(t) \leq \frac{C}{R} y_R(t_0).$$

By (6.24), if $N = 4$ or $N = 5$, $\frac{y_R(t_0)}{R} \rightarrow 0$ as $R \rightarrow \infty$ and $2c_0 \leq -y'_R(t_0)$. Thus (6.32) gives an immediate contradiction in this cases. The remaining case $N = 3$, which is the limit case in (6.32) will be treated in Steps 4 and 5.

By Step 2 and the fact that $N \geq 3$ we have, for $t_0 - \varepsilon_0 R \leq t \leq t_0$,

$$(6.33) \quad \begin{aligned} y'_R(t)^2 &= 4 \left(\int \varphi_R W^+ \partial_t W^+ \right)^2 \\ &\leq 4 \int \varphi_R (W^+)^2 \int \varphi_R (\partial_t W^+)^2 \leq \frac{N-2}{N-1} y_R(t) \left(y''_R(t) + \frac{C_0}{R} \right). \end{aligned}$$

Claim 6.6 (Differential inequality argument). *Let $T > 0$ and $y \in C^2([0, T])$. Assume*

$$\forall t \in [0, T], \quad y'(t) \geq c_0 > 0, \quad y(t) > 0$$

and for some $C_1 > 0$,

$$(6.34) \quad \forall t \in [0, T], \quad y'(t)^2 \leq \frac{N-2}{N-1} y(t) \left[y''(t) + \frac{C_1}{T} \right].$$

Then there is a constant $C > 0$ (depending only on N , c_0 and C_1 , but not on T , such that

$$\forall t \in \left[0, \frac{T}{2}\right], \quad T y'(t) \leq C y(0).$$

Proof. By (6.34),

$$\frac{y'(t)}{y(t)} \leq \frac{N-2}{N-1} \left(\frac{y''(t)}{y'(t)} + \frac{C_1}{T y'(t)} \right) \leq \frac{N-2}{N-1} \left(\frac{y''(t)}{y'(t)} + \frac{C_1}{T c_0} \right).$$

Then, integrating between s and t , $0 \leq s \leq t \leq T$,

$$\log \frac{y(t)}{y(s)} \leq \frac{N-2}{N-1} \left(\log \frac{y'(t)}{y'(s)} + \frac{C_1}{c_0} \right), \text{ i.e. } e^{-\frac{C_1}{c_0}} \frac{y'(s)}{y(s)^{\frac{N-1}{N-2}}} \leq \frac{y'(t)}{y(t)^{\frac{N-1}{N-2}}}.$$

Integrating with respect to t between s and T , we get

$$e^{-\frac{C_1}{c_0}} \frac{y'(s)}{y(s)^{\frac{N-1}{N-2}}} (T-s) \leq -(N-2) \left(\frac{1}{y(T)^{\frac{1}{N-2}}} - \frac{1}{y(s)^{\frac{1}{N-2}}} \right) \leq \frac{N-2}{y(s)^{\frac{1}{N-2}}},$$

which yields

$$(6.35) \quad \forall s \in [0, T], \quad \frac{y'(s)}{y(s)} (T-s) \leq (N-2) e^{\frac{C_1}{c_0}} \leq C.$$

Integrating (6.35) between 0 and $t \in [0, \frac{T}{2}]$ we get

$$\log(y(t)) \leq \log(y(0)) + C \log \left(\frac{T}{T-t} \right) \leq \log(y(0)) + C \log 2, \text{ i.e. } y(t) \leq C y(0),$$

which, gives together with (6.35), the announced result. \square

By (6.33) and the preceding claim on the function $y = y_R(t_0 - t)$, with R large, $T = \varepsilon_0 R$ and $C_1 = C_0 \varepsilon_0$, we get (6.32).

Step 4. Let us show that there exists a constant $C > 0$ such that

$$(6.36) \quad \forall R \geq 1, \quad \forall t \in \mathbb{R}, \quad |y_R''(t)| \leq C d(t).$$

Indeed

$$(6.37) \quad y_R''(t) = 8 \int (\partial_t W^+)^2 \varphi_R + 2 \int (|\nabla W^+|^2 - |W^+|^6 - |\partial_t W^+|^2) \varphi_R + \int (W^+)^2 \Delta \varphi_R.$$

Let us fix $t \in \mathbb{R}$. First assume that $d(t) \leq \delta_0$. Then by Lemma 3.7, there exists λ_0, x_0 such that $\|\nabla(W^+(t) - W_{\lambda_0, x_0})\|_2 + \|\partial_t W^+(t)\|_2 \leq C d(t)$. Noting that

$$8 \int (\partial_t W_{\lambda_0, x_0})^2 \varphi_R + 2 \int (|\nabla W_{\lambda_0, x_0}|^2 - |W_{\lambda_0, x_0}|^6) \varphi_R + \int (W_{\lambda_0, x_0})^2 \Delta \varphi_R = 0,$$

we get (6.36) for $d(t) \leq \delta_0$ by developping (6.37).

Now assume that $d(t) \geq \delta_0$. Thus $\frac{2\|\nabla W\|_2^2}{\delta_0}d(t_0) \geq \|\nabla W\|_2^2$. As a consequence,

$$\left(1 + \frac{2\|\nabla W\|_2^2}{\delta_0}\right)d(t) \geq \|\nabla W^+\|_2^2 + \|\partial_t W^+\|_2^2.$$

By the energy equality $E(W^+, \partial_t W^+) = E(W, 0)$, $\|W^+\|_6^6 = 3\|\nabla W^+\|_2^2 + 3\|\partial_t W^+\|_2^2 + E(W, 0)$. Thus there exists a constant $C > 0$ such that

$$d(t_0) \geq \delta_0 \implies Cd(t) \geq \|\nabla W^+\|_2^2 + \|W^+\|_6^6 + \|\partial_t W^+\|_2^2,$$

which shows, together with (6.37), implies estimate (6.36) in the case $d(t) \geq \delta_0$

Step 5. End of the proof in the case $N = 3$. Let us first show

$$(6.38) \quad \int_{-\infty}^{+\infty} d(t)dt < \infty.$$

Estimates (6.24) and (6.32) give a constant C , independent of R , such that $0 \leq -y'_R(t_0 - \varepsilon_0 R) \leq C$. Thus by (6.23) and (6.28),

$$\begin{aligned} \int_{t_0 - \varepsilon_0 R}^{+\infty} d(t)dt &= \int_{t_0 - \varepsilon_0 R}^{t_0} d(t)dt + \int_{t_0}^{+\infty} d(t)dt \leq \int_{t_0 - \varepsilon_0 R}^{t_0} \left(y''_R(t) + \frac{C}{R}\right)dt + \int_{t_0}^{+\infty} y''_R(t)dt \\ &\leq C\varepsilon_0 + \int_{t_0 - \varepsilon_0 R}^{+\infty} y''_R(t)dt = C\varepsilon_0 - y'_R(t_0 - \varepsilon_0 R) \leq C. \end{aligned}$$

Letting R tends to ∞ we get (6.38).

Let $M \gg 1$. Let $t \in [t_0 - M, t_0]$. By (6.28), for $R \geq R_0$, $y''_R(t) \geq 4d(t) - \frac{C_0}{R}$. Taking $R_M \gg 1$ so that $\min_{t_0 - M \leq t \leq t_0} d(t) \geq \frac{C_0}{R_M}$, we get that $y''_{R_M}(t) \geq 2d(t)$ for $t_0 - M \leq t \leq t_0$ and thus, in view of (6.23),

$$(6.39) \quad \forall t \geq t_0 - M, \quad y''_{R_M}(t) \geq 2d(t).$$

By (6.38), there exists a sequence $t_n \rightarrow -\infty$ such that $d(t_n) \rightarrow 0$. As a consequence, $y'_R(t_n) \rightarrow 0$. We have $y'_R(t_n) = -\int_{t_n}^{t_0 - M} y''_R(t)dt - \int_{t_0 - M}^{+\infty} y''_R(t)dt$, which yields by (6.36) and (6.39)

$$\int_{t_0 - M}^{+\infty} d(t)dt \leq C \int_{t_n}^{t_0 - M} d(t)dt + \frac{1}{2}|y'_R(t_n)|.$$

Letting n tends to infinity, we obtain

$$\forall M \gg 1, \quad \int_{t_0 - M}^{+\infty} d(t)dt \leq C \int_{-\infty}^{t_0 - M} d(t)dt.$$

Note that by (6.38), both integrals in the preceding inequality are finite. Letting M tends to $+\infty$, we get $\int_{-\infty}^{+\infty} d(t)dt = 0$. This shows that $d(t) = 0$ for all t , which is a contradiction. \square

APPENDIX A. ESTIMATES ON THE MODULATION PARAMETERS

In this appendix we prove Lemma 3.7. *Proof of (3.19).* In the proof of estimate (3.19), t is just a parameter that we will systematically omit.

Developping the equality $E((1 + \alpha)W + \tilde{f}, \partial_t u) = E(W, 0)$, we get, with (3.13),

$$(A.1) \quad Q(\alpha W + \tilde{f}) + \frac{1}{2}\|\partial_t u\|_2^2 = O(\|\nabla(\alpha W + \tilde{f})\|_2^3).$$

By the orthogonality of \tilde{f} with W in \dot{H}^1 , and the equation $\Delta W + W^{\frac{N+2}{N-2}} = 0$ we have

$$\int \nabla W \cdot \nabla \tilde{f} = \int W^{\frac{N+2}{N-2}} \tilde{f} = 0.$$

Thus W and \tilde{f} are Q -orthogonal and $Q(\alpha W + \tilde{f}) = -\alpha^2 |Q(W)| + Q(\tilde{f})$. Hence

$$(A.2) \quad -\alpha^2 |Q(W)| + Q(\tilde{f}) + \frac{1}{2} \|\partial_t u\|_2^2 = O(\|\nabla(\alpha W + \tilde{f})\|_2^3).$$

Now

$$(A.3) \quad \|\nabla(\alpha W + \tilde{f})\|_2^2 = \alpha^2 \|\nabla W\|_2^2 + \|\nabla \tilde{f}\|_2^2.$$

If δ_0 is small, so is $\|\nabla(\alpha W + \tilde{f})\|_{L^2}$, so that by (A.2) and Claim 3.5, there exists $c > 0$ such that $\alpha^2 \geq c \left(\|\nabla \tilde{f}\|_2^2 + \|\partial_t u\|_2^2 \right)$. Thus by (A.3), $\|\nabla(\alpha W + \tilde{f})\|_2^2 \approx \alpha^2$. Using again (A.2), we get

$$\alpha^2 \approx \|\nabla \tilde{f}\|_2^2 + \|\partial_t u\|_2^2.$$

We have

$$\|\nabla u\|_2^2 - \|\nabla W\|_2^2 = \|\nabla(W + \alpha W + \tilde{f})\|_2^2 - \|\nabla W\|_2^2 = (2\alpha + \alpha^2) \|\nabla W\|_2^2 + \|\nabla \tilde{f}\|_2^2 \approx \alpha,$$

which concludes the proof of (3.19).

Proof of (3.20). Let $v(t) := u_{\mu, X}(t)$. Then

$$(A.4) \quad u = \mu^{\frac{N-2}{2}} v(t, \mu(x - X(t))).$$

Differentiating (A.4), and writing $y = \mu(x - X(t))$, we get

$$\partial_t u(t, x) = \left(\frac{N}{2} - 1 \right) \frac{\mu'}{\mu} u(t, x) + \mu^{\frac{N-2}{2}} \left[\mu'(x - X(t)) \cdot \nabla_y v(t, y) - \mu X' \cdot \nabla v(t, y) + \partial_t v(t, y) \right].$$

Recall that $v = W + \alpha W + \tilde{f}$. Multiplying the preceding equation by $\mu^{-\frac{N}{2}}$, we obtain

$$\begin{aligned} w(t, y) &:= \frac{1}{\mu^{\frac{N}{2}}} \partial_t u\left(t, \frac{y}{\mu} + X\right) \\ &= \left(\frac{N-2}{2} \right) \frac{\mu'}{\mu^2} (W + \alpha W + \tilde{f}) + \frac{\mu'}{\mu^2} y \cdot \nabla_y (W + \alpha W + \tilde{f}) - X' \cdot \nabla_y (W + \alpha W + \tilde{f}) + \frac{1}{\mu} \frac{\partial}{\partial t} (\alpha W + \tilde{f}). \end{aligned}$$

Hence

$$\begin{aligned} w &= -\frac{\mu'}{\mu^2} \widetilde{W} - \sum_{j=1}^N X'_j(t) W_j + \frac{\alpha'}{\mu} W + (R) \\ (R) &:= \frac{\mu'}{\mu^2} \left(\frac{N-2}{2} + y \cdot \nabla_y \right) (\alpha W + \tilde{f}) - \sum_{j=1}^N x'_j(t) \partial_{y_j} (\alpha W + \tilde{f}) + \frac{1}{\mu} \partial_t \tilde{f}. \end{aligned}$$

Hence (using the orthogonality in \dot{H}^1 of W , \widetilde{W} , W_1, \dots, W_N)

$$(A.5) \quad \begin{cases} \int w \Delta W = -\frac{\alpha'}{\mu} \|\nabla W\|_2^2 + \int (R) \Delta W \\ \int w \Delta \widetilde{W} = -\frac{\mu'}{\mu^2} \|\nabla \widetilde{W}\|_2^2 + \int (R) \Delta \widetilde{W} \\ \int w \Delta W_j = -x'_j(t) \|\nabla W_j\|_2^2 + \int (R) \Delta W_j, \quad j = 1 \dots N. \end{cases}$$

Let

$$\mathbf{d}_1(t) := \mathbf{d}(t) + \left| \frac{\alpha'}{\mu} \right| + \left| \frac{\mu'}{\mu^2} \right| + |X'(t)|.$$

Then, noting that for all t , $\partial_t \widetilde{f}(t) \in H^\perp$ and using estimate (3.19), we have

$$\left| \int (R) \Delta W \right| + \left| \int (R) \Delta (-iW) \right| + \left| \int (R) \Delta \widetilde{W} \right| + \sum_{j=1}^N \left| \int (R) \Delta W_j \right| \leq C \mathbf{d}_1^2(t).$$

Summing up estimates (A.5) and using that $\|w(t)\|_2 = \|\partial_t u(t)\|_2 \lesssim \mathbf{d}(t)$, we get

$$\mathbf{d}_1(t) \leq \mathbf{d}_1^2(t) + \mathbf{d}(t)$$

which yields (3.20) for small $\mathbf{d}(t)$. \square

APPENDIX B. SOME TECHNICAL ESTIMATES

In this appendix we proof Lemma 5.1.

We will need the following version of Hölder estimate for fractional Sobolev spaces

Claim B.1. *Let $p, p_2, p_3, p_4 \in (1, \infty)$, $p_1 \in (1, \infty]$, such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, and $\frac{1}{p_3} + \frac{1}{p_4} = \frac{1}{p}$.*

(a) $\|D_x^{1/2}(fg)\|_p \lesssim \|f\|_{p_1} \|D_x^{1/2}g\|_{p_2} + \|D_x^{1/2}f\|_{p_3} \|g\|_{p_4}$.

(b) *Let $F \in C^1(\mathbb{R})$ such that $F(0) = 0$. Then*

$$\|D_x^{1/2}F(f)\|_p \lesssim \|F'(f)\|_{p_1} \|D_x^{1/2}f\|_{p_2}$$

(c) *Let $m \in \mathbb{N}^*$ and $r_1, r_2, r_3 \in (1, \infty)$ such that $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = 1$. Then*

$$\begin{aligned} \|D_x^{1/2}(f^{2m} - g^{2m})\|_p &\lesssim \left(\|f^{2m-1}\|_{p_1} + \|g^{2m-1}\|_{p_1} \right) \|D_x^{1/2}(f - g)\|_{p_2} \\ &\quad + \left(\|f^{2m-2}\|_{r_1} + \|g^{2m-2}\|_{r_1} \right) \left(\|D_x^{1/2}f\|_{r_2} + \|D_x^{1/2}g\|_{r_2} \right) \|f - g\|_{r_3}. \end{aligned}$$

Points (a) and (b) follows from [KPV93, Theorems A.7, A.8, A.9 and A.12]. Point (c) is a consequence of (a).

Proof of (5.2).

By Claim B.1 (a), we have

$$(B.1) \quad \left\| D_x^{1/2} \left(u(t) W^{\frac{4}{N-2}} \right) \right\|_{\frac{2(N+1)}{N+3}} \leq C \left\{ \left\| D_x^{1/2} u(t) \right\|_{\frac{2(N+1)}{N-1}} \left\| W^{\frac{4}{N-2}} \right\|_{\frac{N+1}{2}} + \|u(t)\|_{\frac{2(N+1)}{N-2}} \left\| D_x^{1/2} \left(W^{\frac{4}{N-2}} \right) \right\|_{\frac{2(N+1)}{5}} \right\}.$$

Furthermore, W is a C^∞ function on \mathbb{R}^N such that $W^{\frac{4}{N-2}} \approx 1/|x|^4$, $D_x \left(W^{\frac{4}{N-2}} \right) \approx 1/|x|^5$. Hence $W^{\frac{4}{N-2}}$ belongs to $L^{\frac{N+1}{2}} \cap W^{1, \frac{2(N+1)}{5}}$. As a consequence we can rewrite (B.1) as

$$\left\| D_x^{1/2} \left(u(t) W^{\frac{4}{N-2}} \right) \right\|_{\frac{2(N+1)}{N+3}} \leq C \left(\left\| D_x^{1/2} u(t) \right\|_{\frac{2(N+1)}{N-1}} + \left\| u(t) \right\|_{\frac{2(N+1)}{N-2}} \right),$$

which gives (5.2), using Hölder inequality in time.

We will skip the proof of (5.3) which is a direct consequence of Hölder inequality.

Proof of (5.4).

Fixing t , we will show

(B.2)

$$\begin{aligned} \left\| D_x^{1/2} (R(u) - R(v)) \right\|_{\frac{2(N+1)}{N+3}} &\lesssim \left\| D_x^{1/2} (u - v) \right\|_{\frac{2(N+1)}{N-1}} \left[\left\| |u| + |v| \right\|_{\frac{2(N+1)}{N-2}} + \left\| |u| + |v| \right\|_{\frac{4}{N-2}}^{\frac{2(N+1)}{N-2}} \right] \\ &\quad + \left\| u - v \right\|_{\frac{2(N+1)}{N-2}} \left[\left\| |u| + |v| \right\|_{\frac{2(N+1)}{N-2}} + \left\| |u| + |v| \right\|_{\frac{4}{N-2}}^{\frac{2(N+1)}{N-2}} \right] \\ &\quad + \left\| |D_x^{1/2} u| + |D_x^{1/2} v| \right\|_{\frac{2(N+1)}{N-1}} + \left\| |D_x^{1/2} u| + |D_x^{1/2} v| \right\|_{\frac{4}{N-1}}^{\frac{2(N+1)}{N-1}} \right]. \end{aligned}$$

Hölder inequality in time will yield the desired result. We have $R(u) = W^{\frac{N+2}{N-2}} J(W^{-1}u)$, where $J(s) = |1 + s|^{\frac{4}{N-2}}(1 + s) - 1 - \frac{N+2}{N-2}s$, $J'(s) = \frac{N+2}{N-2}|1 + s|^{\frac{4}{N-2}} - \frac{N+2}{N-2}$. Hence

$$(B.3) \quad R(u) - R(v) = \frac{N+2}{N-2} (u - v) \underbrace{\int_0^1 \left(|W + v + (u - v)\theta|^{\frac{4}{N-2}} - W^{\frac{4}{N-2}} \right) d\theta}_{I(u, v)}.$$

Hence, by Claim B.1 (a),

$$\left\| D_x^{1/2} (R(u) - R(v)) \right\|_{\frac{2(N+1)}{N+3}} \lesssim \left\| D_x^{1/2} (u - v) \right\|_{\frac{2(N+1)}{N-1}} \|I(u, v)\|_{\frac{N+1}{2}} + \|u - v\|_{\frac{2(N+1)}{N-2}} \left\| D_x^{1/2} I(u, v) \right\|_{\frac{2(N+1)}{5}}.$$

In view of the integral expression of $I(u, v)$, (B.2) will follow from the estimates

$$(B.4) \quad \left\| |W + h|^{\frac{4}{N-2}} - W^{\frac{4}{N-2}} \right\|_{\frac{N+1}{2}} \lesssim \|h\|_{\frac{2(N+1)}{N-2}} + \|h\|_{\frac{4}{N-2}}^{\frac{2(N+1)}{N-2}}$$

$$(B.5) \quad \begin{aligned} \left\| D_x^{1/2} \left(|W + h|^{\frac{4}{N-2}} - W^{\frac{4}{N-2}} \right) \right\|_{\frac{2(N+1)}{5}} \\ \lesssim \|h\|_{\frac{2(N+1)}{N-2}} + \|h\|_{\frac{4}{N-2}}^{\frac{2(N+1)}{N-2}} + \|D_x^{1/2} h\|_{\frac{2(N+1)}{N-1}} + \|D_x^{1/2} h\|_{\frac{4}{N-1}}^{\frac{2(N+1)}{N-1}}. \end{aligned}$$

Let us first show (B.4). By the pointwise bound $\left| |W + h|^{\frac{4}{N-2}} - |W|^{\frac{4}{N-2}} \right| \lesssim W^{\frac{6-N}{N-2}} |h| + |h|^{\frac{4}{N-2}}$, and Hölder inequality

$$\left\| |W + h|^{\frac{4}{N-2}} - W^{\frac{4}{N-2}} \right\|_{\frac{N+1}{2}} \lesssim \|W^{\frac{6-N}{N-2}}\|_{\frac{2(N+1)}{6-N}} \|h\|_{\frac{2(N+1)}{N-2}} + \|h\|_{\frac{4}{N-2}}^{\frac{2(N+1)}{N-2}}.$$

Noting that $\|W^{\frac{6-N}{N-2}}\|_{\frac{2(N+1)}{6-N}} = \|W\|_{\frac{2(N+1)}{N-2}}^{\frac{6-N}{N-2}} < \infty$, we get (B.4). It remains to show (B.5). We will distinguish two cases.

First case: $N = 3$ or $N = 4$. Then $\frac{4}{N-2} \in \{2, 4\}$. By Claim B.1, (c),

$$\begin{aligned} & \left\| D_x^{1/2} \left((W+h)^{\frac{4}{N-2}} - W^{\frac{4}{N-2}} \right) \right\|_{\frac{2(N+1)}{5}} \\ & \lesssim \left(\left\| (W+h)^{\frac{6-N}{N-2}} \right\|_{\frac{2(N+1)}{6-N}} + \left\| h^{\frac{6-N}{N-2}} \right\|_{\frac{2(N+1)}{6-N}} \right) \left\| D_x^{1/2} h \right\|_{\frac{2(N+1)}{N-1}} \\ & \quad + \left(\left\| (W+h)^{\frac{8-2N}{N-2}} \right\|_{\frac{2(N+1)}{8-2N}} + \left\| h^{\frac{8-2N}{N-2}} \right\|_{\frac{2(N+1)}{8-2N}} \right) \\ & \quad \times \left(\left\| D_x^{1/2} (W+h) \right\|_{\frac{2(N+1)}{N-1}} + \left\| D_x^{1/2} h \right\|_{\frac{2(N+1)}{N-1}} \right) \left\| h \right\|_{\frac{2(N+1)}{N-2}}. \end{aligned}$$

Hence

$$\begin{aligned} & \left\| D_x^{1/2} \left((W+h)^{\frac{4}{N-2}} - W^{\frac{4}{N-2}} \right) \right\|_{\frac{2(N+1)}{5}} \lesssim \left(1 + \left\| h \right\|_{\frac{2(N+1)}{N-2}} \right)^{\frac{6-N}{N-2}} \left\| D_x^{1/2} h \right\|_{\frac{2(N+1)}{N-1}} \\ & \quad + \left(1 + \left\| h \right\|_{\frac{8-2N}{N-2}} \right) \left(1 + \left\| D_x^{1/2} h \right\|_{\frac{2(N+1)}{N-1}} \right) \left\| h \right\|_{\frac{2(N+1)}{N-2}} \\ & \lesssim \left\| h \right\|_{\frac{2(N+1)}{N-2}} + \left\| h \right\|_{\frac{4}{N-2}}^{\frac{4}{N-2}} + \left\| D_x^{1/2} h \right\|_{\frac{2(N+1)}{N-1}} + \left\| D_x^{1/2} h \right\|_{\frac{4}{N-2}}^{\frac{4}{N-2}}, \end{aligned}$$

by the convexity inequality $AB \leq \frac{6-N}{4} A^{\frac{4}{6-N}} + \frac{N-2}{4} B^{\frac{4}{N-2}}$. This yields (B.5), and concludes the proof of (B.2) (thus of (5.4)) when $N = 3$ or $N = 4$.

Second case: $N = 5$.

In this case, we must bound $\left\| D_x^{1/2} (|W+h|^{4/3} - W^{4/3}) \right\|_{\frac{12}{5}}$ by sum of powers of $\|h\|_4$ and $\|D_x^{1/2} h\|_3$. Note that Claim B.1 (c) is no longer available. We have

$$(B.6) \quad |W+h|^{4/3} - W^{4/3} = W^{4/3} F(W^{-1}h) + |h|^{4/3}, \quad F(s) = |1+s|^{4/3} - 1 - |s|^{4/3}.$$

By Claim B.1, (b)

$$(B.7) \quad \left\| D_x^{1/2} |h|^{4/3} \right\|_{\frac{12}{5}} \lesssim \left\| |h|^{1/3} \right\|_{12} \left\| D_x^{1/2} h \right\|_3 \lesssim \left\| h \right\|_4^{1/3} \left\| D_x^{1/2} h \right\|_3 \lesssim \left\| h \right\|_4^{4/3} + \left\| D_x^{1/2} h \right\|_3^{4/3},$$

by the convexity inequality $AB \leq \frac{3}{4} A^{4/3} + \frac{1}{4} B^4$.

Note that F is C^1 and that F' is bounded. In order to apply Claim B.1 (b) to $W^{4/3} F(W^{-1}h)$ we will need a dyadic decomposition of \mathbb{R}^5 . Let $\varphi \in C^\infty(\mathbb{R}^5)$ such that $\varphi(x) = 1$ if $|x| \leq 1$ and $\varphi(x) = 0$ if $|x| \geq 2$. Define $\psi(x) := \varphi(x/2) - \varphi(x)$, so that $\text{supp } \psi \subset \{1 \leq |x| \leq 4\}$. Let $\psi_k(x) := \varphi(x/2^{k-1})$ for $k \geq 1$ and $\psi_0(x) := \varphi(x)$. Then

$$\text{supp } \psi_k \subset \left\{ \frac{1}{2^{k-1}} \leq |x| \leq \frac{1}{2^{k+1}} \right\}, \quad k \geq 1, \quad \text{supp } \psi_0 \subset \{|x| \leq 2\}; \quad \sum_{k \geq 0} \psi_k(x) = 1.$$

Choose also $\tilde{\psi} \in C_0^\infty(\mathbb{R}^5)$ such that $\text{supp } \tilde{\psi} \subset \{1/2 \leq |x| \leq 8\}$ and $\tilde{\psi}(x) = 1$ on $\{2 \leq |x| \leq 4\}$. Let $\tilde{\psi}_k(x) := \tilde{\psi}(x/2^{k-1})$ for $k \geq 1$ and $\psi_0(x) := \varphi(x/2)$. Then

$$\text{supp } \tilde{\psi}_k \subset \left\{ \frac{1}{2^{k-2}} \leq |x| \leq \frac{1}{2^{k+2}} \right\}, \quad k \geq 1, \quad \text{supp } \tilde{\psi}_0 \subset \{|x| \leq 4\}; \quad x \in \text{supp } \tilde{\psi}_k \implies \psi_k(x) = 1.$$

We have

$$(B.8) \quad W^{4/3}F(W^{-1}h) = \sum_{k \geq 0} \psi_k W^{4/3}F(W^{-1}h) = \sum_{k \geq 0} \psi_k W^{4/3}F\left(W^{-1}\tilde{\psi}_k(x)h\right).$$

We leave the proof of the following estimates which follow from the explicit expression of W and scaling arguments to the reader.

Claim B.2. *For all $p \in [1, \infty]$ and for all $k \geq 0$,*

$$\begin{aligned} \left\| \psi_k W^{4/3} \right\|_p &\lesssim 2^{(-4+5/p)k}, & \left\| D_x^{1/2}(\psi_k W^{4/3}) \right\|_p &\lesssim 2^{(-9/2+5/p)k}, \\ \left\| \tilde{\psi}_k W^{-1} \right\|_p &\lesssim 2^{(3+5/p)k}, & \left\| D_x^{1/2}(\tilde{\psi}_k W^{-1}) \right\|_p &\lesssim 2^{(5/2+5/p)k}. \end{aligned}$$

By Claim B.1 (a),

$$(B.9) \quad \begin{aligned} &\left\| D_x^{1/2}(\psi_k W^{4/3}F(W^{-1}\tilde{\psi}_k h)) \right\|_{\frac{12}{5}} \\ &\lesssim \left\| D_x^{1/2}(\psi_k W^{4/3}) \right\|_6 \left\| F(W^{-1}\tilde{\psi}_k h) \right\|_4 + \left\| \psi_k W^{4/3} \right\|_{12} \left\| D_x^{1/2}(F(W^{-1}\tilde{\psi}_k h)) \right\|_3. \end{aligned}$$

Note that $|F(s)| \lesssim |s|$, so that, in view of Claim B.2,

$$\left\| F(W^{-1}\tilde{\psi}_k h) \right\|_4 \lesssim \left\| W^{-1}\tilde{\psi}_k h \right\|_4 \lesssim 2^{3k} \|h\|_4$$

Thus by Claim B.2 again,

$$(B.10) \quad \left\| D_x^{1/2}(\psi_k W^{4/3}) \right\|_6 \left\| F(W^{-1}\tilde{\psi}_k h) \right\|_4 \lesssim 2^{(-9/2+5/6+3)k} = 2^{-\frac{2}{3}k} \|h\|_4.$$

Furthermore, F' being bounded, by Claim B.1 (b), then (a)

$$\begin{aligned} \left\| D_x^{1/2}(F(W^{-1}\tilde{\psi}_k h)) \right\|_3 &\lesssim \left\| D_x^{1/2}(W^{-1}\tilde{\psi}_k h) \right\|_3 \\ &\lesssim \left\| D_x^{1/2}(W^{-1}\tilde{\psi}_k) \right\|_{12} \|h\|_4 + \left\| W^{-1}\tilde{\psi}_k \right\|_\infty \left\| D_x^{1/2}h \right\|_3. \end{aligned}$$

Thus by Claim B.2,

$$(B.11) \quad \begin{aligned} \left\| \psi_k W^{4/3} \right\|_{12} \left\| D_x^{1/2}(F(W^{-1}\tilde{\psi}_k h)) \right\|_3 &\lesssim 2^{(-4+5/12)k} \left(2^{(3+5/12)k} \|h\|_4 + 2^{3k} \|D_x^{1/2}h\|_3 \right) \\ &\lesssim 2^{-\frac{1}{6}k} \|h\|_4 + 2^{-\frac{7}{12}k} \|D_x^{1/2}h\|_3. \end{aligned}$$

By (B.9), (B.10) and (B.11),

$$(B.12) \quad \left\| D_x^{1/2}(W^{4/3}F(W^{-1}h)) \right\|_{\frac{12}{5}} \lesssim \sum_{k \geq 0} \left\| D_x^{1/2}(\psi_k W^{4/3}F(W^{-1}\tilde{\psi}_k h)) \right\|_{\frac{12}{5}} \lesssim \|h\|_4 + \|D_x^{1/2}h\|_3.$$

In view of (B.6), we get, by (B.7) and (B.12)

$$\left\| D_x^{1/2}(|W+h|^{4/3} - W^{4/3}) \right\|_{\frac{12}{5}} \lesssim \|h\|_4^{4/3} + \|D_x^{1/2}h\|_3^{4/3} + \|h\|_4 + \|D_x^{1/2}h\|_3.$$

This yields (B.5), concluding the proof of (5.4) in the case $N = 5$. \square

APPENDIX C. DERIVATIVE OF g_R

Claim C.1. *Let u be a solution of (1.1) such that $E(u_0, u_1) = E(W, 0)$ and g_R be defined by (3.7). There exist C^∞ real-valued functions on \mathbb{R}^N , a_R^{jk} , b_R^1 , b_R^2 , b_R^3 , supported in $\{|x| \geq R\}$, bounded independently of R and such that*

$$g'_R(t) = \frac{1}{N-2} \int |\partial_t u|^2 dx - \frac{1}{N-2} \left(\int |\nabla W|^2 dx - \int |\nabla u|^2 dx \right) + A_R(u, \partial_t u).$$

where

$$(C.1) \quad A_R(u, \partial_t u) := \sum_{j,k} \int a_R^{jk} \partial_j u \partial_k u dx + \int b_R^1 (\partial_t u)^2 + b_R^2 u^{2^*} + \frac{1}{|x|^2} b_R^3 u^2 dx.$$

Proof.

$$(C.2) \quad \frac{d}{dt} \int \psi_R \cdot \nabla u \partial_t u dx = \int \psi_R \cdot \nabla u \left(\Delta u + |u|^{\frac{N+2}{N-2}} u \right) dx + \int \psi_R \cdot \nabla \partial_t u \partial_t u dx.$$

Furthermore, denoting by ψ_{Rj} , $j = 1 \dots N$, the coordinates of ψ_R ,

$$\begin{aligned} \int \psi_R \cdot \nabla u (\Delta u) dx &= \sum_{j,k} \int \psi_{Rj} \partial_j u \partial_k^2 u dx \\ &= -\frac{1}{2} \sum_j \int \frac{\partial \psi_{Rj}}{\partial x_j} (\partial_j u)^2 dx + \frac{1}{2} \sum_{\substack{j,k \\ j \neq k}} \int \frac{\partial \psi_{Rj}}{\partial x_k} (\partial_j u)^2 dx - \sum_{\substack{j,k \\ j \neq k}} \int \frac{\partial \psi_{Rj}}{\partial x_k} \partial_j u \partial_k u dx. \end{aligned}$$

Note that if $|x| \leq R$, $\frac{\partial \psi_{Rj}}{\partial x_j}(x) = 1$ and $\frac{\partial \psi_{Rj}}{\partial x_k}(x) = 0$ ($j \neq k$). Thus

$$\int \psi_R \cdot \nabla u (\Delta u) dx = \frac{N-2}{2} \int |\nabla u|^2 dx + \sum_{j,k} \int_{|x| \geq R} \tilde{a}_{Rjk} \partial_j u \partial_k u dx,$$

where the \tilde{a}_{Rjk} are bounded independently of R , C^∞ , and supported in $|x| \geq R$. By similar integration by parts on the other terms of (C.2), we get

$$(C.3) \quad \frac{d}{dt} \int \psi_R(x) \cdot \nabla u \partial_t u = \frac{N-2}{2} \int |\nabla u|^2 dx - \frac{N-2}{2} \int |u|^{2^*} dx - \frac{N}{2} \int (\partial_t u)^2 dx + \tilde{A}_R(u, \partial_t u),$$

where $\tilde{A}_R(u, \partial_t u)$ is a sum of integrals of the desired form (C.1).

Furthermore,

$$\begin{aligned} \frac{N-1}{2} \frac{d}{dt} \int \varphi_R u \partial_t u dx &= \frac{N-1}{2} \int \varphi_R (\partial_t u)^2 + \varphi_R u \left(\Delta u + |u|^{\frac{N+2}{N-2}} u \right) dx \\ &= \frac{N-1}{2} \int \varphi_R (\partial_t u)^2 - \varphi_R |\nabla u|^2 + \frac{1}{2} (\Delta \varphi_R) u^2 + \varphi_R |u|^{2^*} dx. \end{aligned}$$

Noting that for $|x| \leq R$, $\varphi_R = 1$ and $\Delta \varphi_R = 0$, and that if $|x| \geq 2R$, $\Delta \varphi_R = 0$, we get

$$(C.4) \quad \frac{N-1}{2} \frac{d}{dt} \int \varphi_R u \partial_t u dx = \frac{N-1}{2} \int -|\nabla u|^2 + |u|^{2^*} + (\partial_t u)^2 dx + \hat{A}_R(u, \partial_t u),$$

where $\hat{A}_R(u, \partial_t u)$ is again of the form (C.1).

Summing up (C.3) and (C.4), we obtain

$$g'_R(t) = -\frac{1}{2} \int |\nabla u|^2 dx + \frac{1}{2} \int |u|^{2^*} dx - \frac{1}{2} \int (\partial_t u)^2 dx + A_R(u, \partial_t u),$$

where A_R is defined by (C.1) for some functions a_R^{jk} , b_R^1 , b_R^2 , b_R^3 . To conclude the proof, note that $E(u, \partial_t u) = E(W, 0)$ implies

$$\frac{1}{2} \int |u|^{2^*} = \frac{N}{2(N-2)} \int |\nabla u|^2 dx + \frac{N}{2(N-2)} \int (\partial_t u)^2 dx - \frac{N}{N-2} E(W, 0),$$

and recall that $E(W, 0) = \frac{1}{N} \|\nabla W\|_2^2$, which gives (C.2). \square

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